

# POSITIVITY PRESERVING INTERPOLATION BY USING RATIONAL CUBIC BALL SPLINE

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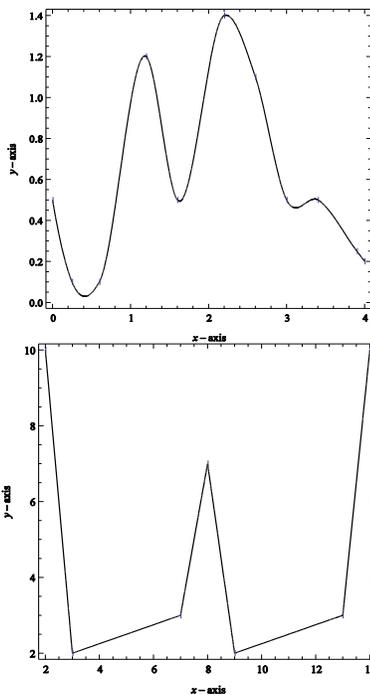
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## Graphical abstract



## Abstract

This paper discusses the positivity preserving by using rational cubic Ball interpolant of the form cubic/quadratic with two parameters. The sufficient condition for the positivity is derived on one parameter meanwhile the other one is a free parameter to control the final shape of the interpolating curves. The degree smoothness achieved is  $C^1$ . From numerical results, the rational cubic Ball spline with two parameters gives smooth interpolating positive curves as well as visually pleasing for computer graphics visualization. Furthermore the scheme is better than existing schemes i.e. its easiness to use and less computation. All numerical results are produced by using Mathematica.

Keywords: Rational cubic Ball interpolant, positivity, dependent, visualization

## Abstrak

Makalah ini membincangkan pengekalan kepositifan dengan menggunakan penginterpolasi kubik nisbah Ball dalam bentuk kubik/kuadratik dengan dua parameter. Syarat cukup untuk kepositifan akan dihasilkan pada satu parameter manakala yang lagi satu adalah parameter bebas untuk mengawal bentuk akhir lengkung interpolasi. Darjah keselajaran yang dicapai adalah  $C^1$ . Daripada keputusan berangka, kubik splin nisbah Ball dengan dua parameter memberikan lengkung interpolasi yang positif serta gambaran jelas untuk grafik komputer. Disamping itu skema ini lebih baik daripada skema sedia ada i.e. mudah untuk digunakan dan juga kurang pengiraan. Semua keputusan berangka dihasilkan dengan menggunakan perisian Mathematica.

Kata kunci: Penginterpolasi kubik nisbah Ball, kepositifan, kebergantungan, pemaparan

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## 1.0 INTRODUCTION

Shape preserving approximation and interpolation are important in sciences and engineering based applications. One of the requirements is to preserve the positivity of the data sets. For instance the level of Sodium Hydroxide (NaOH) are always positive and any interpolant must be able to produce the interpolating curves with positivity preserving. The rainfall distributions also show a positive value and any negativity values just simply unrealistic. The

common shape preserving interpolation schemes using rational cubic spline of the form cubic numerator and denominator function can be linear or quadratic or cubic with up to four parameters in the description of the rational cubic interpolant. Recently, several researchers have proposed shape preserving interpolation by using rational cubic Ball interpolant. For instance Karim [14] proposed rational cubic Ball interpolant with two parameters. Meanwhile Karim [15, 16] proposed new rational cubic interpolant with four parameters for positivity and monotonicity preserving interpolation. Karim [17]

discussed the positivity and monotonicity preserving interpolation using rational cubic Ball interpolant (cubic numerator and quadratic denominator) with three parameters. Most recently Jaafar *et al.* [13] discussed the positivity preserving interpolation using rational cubic Ball interpolant of the form cubic numerator and cubic denominator with four parameters. Their work is in line with Karim [15, 16]. Tahat *et al.* [23] also discussed the positivity using different form of rational cubic Ball interpolant with four parameters.

Besides the use of rational cubic Ball interpolant, most researchers have used the rational cubic spline of the form of cubic numerator with linear, quadratic or cubic function as denominator with up to four parameters in the description of the rational cubic interpolant. For instance Abbas *et al.* [1] discussed the positivity preserving interpolation using rational cubic spline (quadratic denominator) with three parameters. The sufficient are derived on one parameter meanwhile the remaining two are free parameters. They claimed that their schemes is  $C^2$ . But their schemes suffer from the fact that it may not generate the positive interpolating curves with  $C^2$  continuity. Brodlie and Butt [2] and Butt and Brodlie [5] discussed the convexity and positivity preserving interpolation using cubic Hermite spline. The positivity are achieved by inserting one or two extra knots along the interval in which shape violation are found. Bordlie *et al.*, (1995) extend the idea in [5] for positive surface interpolation. Brodlie *et al.* [4] also discussed the positivity preserving but using the Shepard interpolation family. The positivity and data constrained are achieved by solving some optimization problem. Hussain *et al.* [8] discussed the positivity and convexity shape preserving with  $C^2$  continuity. In general there are many researchers investigated the positivity preserving interpolation, for instance [9, 10, 11, 12-13, 18-21]. Some good surveys on shape preserving approximation and interpolation can be found in [6], [7] and [24].

In this paper new rational cubic Ball spline interpolant (cubic denominator) with two parameters is constructed for positivity preserving interpolation. This work is inspired by the work of Sarfraz *et al.*, (2013).

We identify several nice features of our rational cubic spline for positivity preserving. It is summarized below:

- (i) A new  $C^1$  rational cubic Ball interpolation (cubic/quadratic) with two parameters has been used for positivity preserving. Meanwhile Jaafar *et al.* [13] utilized the rational cubic Ball interpolant (cubic denominator) with four parameters that appear previously in [15, 16].
- (ii) The rational cubic Ball spline scheme does not required any extra knots but Brodlie and Butt [3], Brodlie *et al.* [4] and Butt and Brodlie [5] requires the extra knots (one or

two) to preserves the positivity of the data.

- (iii) Our rational scheme is based from cubic Ball function while in Hussain *et al.* [10] and Ibraheem *et al.* [12] the interpolant are based from rational trigonometric spline. Since in the definition of cubic Ball, there are two basis functions with degree two (quadratic), the proposed rational cubic Ball may require less computation compare to the rational cubic spline and rational cubic trigonometric spline respectively.
- (iv) The proposed rational cubic Ball is simple to use and are better than the existing schemes such as Jaafar *et al.* [13] and Tahat *et al.* [13] – in the sense of easiness to use and visually pleasing of the resulting positive interpolating curves. Furthermore the schemes is not involving any radial basis functions (RBF) as appear in the works of [4] and [25]. Thus no need to find suitable basis functions and sets of parameters by solving any optimization problem.

This paper is organized as follows. The new rational cubic Ball interpolation is discussed in Section 2 including shape control analysis. Arithmetic Mean Method (AMM) formula to estimate the first derivative at the respective knots and the construction of the sufficient conditions for positivity preserving are discussed in details through this Section. Numerical and graphical results for positivity preserving are given in Section 3. A conclusions are given in the final section.

## 2.0 METHODOLOGY

This section introduces the new rational cubic Ball with two parameters  $\alpha_i, \beta_i$   $i = 1, 2, \dots, n-1$ . The shape control of the rational cubic interpolant also will be discussed in details with numerical examples.

### 2.1 Rational Cubic Ball Interpolant

Given that the scalar (or functional) data is given such that  $\{(x_i, f_i), i = 1, 2, \dots, n\}$  where  $x_1 < x_2 < \dots < x_n$ . Choosing  $h_i = x_{i+1} - x_i$ ,  $\Delta_i = (f_{i+1} - f_i)/h_i$  and  $\theta = (x - x_i)/h_i$  where  $0 \leq \theta \leq 1$ . For  $x \in [x_i, x_{i+1}], i = 1, 2, \dots, n-1$  the rational cubic Ball interpolant is defined by

$$S(x) = S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)}, \quad (1)$$

where

$$P_i(\theta) = (1-\theta)^2 \alpha_i f_i + (1-\theta)^2 \theta V_i + (1-\theta) \theta^2 W_i + (1-\theta)^2 \beta_i f_{i+1}$$

$$Q_i(\theta) = \alpha_i + \beta_i (1-\theta)\theta$$

The parameters  $\alpha_i, \beta_i, i = 1, 2, \dots, n-1$  are free parameters. The rational cubic Ball interpolant in (1) satisfies the following  $C^1$  conditions:

$$\begin{aligned} S(x_i) &= f_i, & S^{(1)}(x_i) &= d_i \\ S(x_{i+1}) &= f_{i+1}, & S^{(1)}(x_{i+1}) &= d_{i+1} \end{aligned} \quad (2)$$

From  $C^1$  conditions in (2) the unknown variables  $V_i, W_i, i = 1, 2, \dots, n-1$  is given as follows:

$$V_i = (2\alpha_i + \beta_i)f_i + \alpha_i h_i d_i, \quad W_i = (2\alpha_i + \beta_i)f_{i+1} - \alpha_i h_i d_{i+1}$$

When  $\alpha_i = 1$  and  $\beta_i = 0$ , the rational cubic Ball interpolant defined by (1) is reducing to standard cubic Ball polynomial given as follows:

$$\begin{aligned} S_i(x) &= f_i(1-\theta)^2 + (2f_i + h_i d_i)(1-\theta)^2 \theta + f_{i+1}(1-\theta)^2 \\ &\quad + (2f_{i+1} - h_i d_{i+1})(1-\theta)^2 \theta \end{aligned} \quad (3)$$

Furthermore  $S_i(x)$  can be rewritten as follows:

$$S_i(x) = (1-\theta)f_i + \theta f_{i+1} + \frac{h_i E_i (1-\theta)\theta}{Q_i(\theta)} \quad (4)$$

with

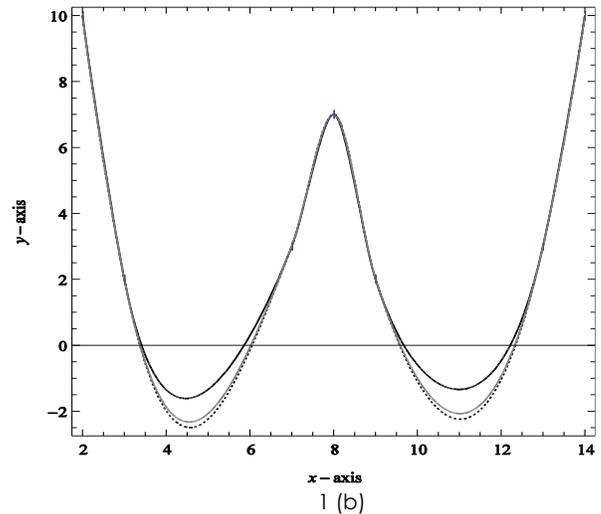
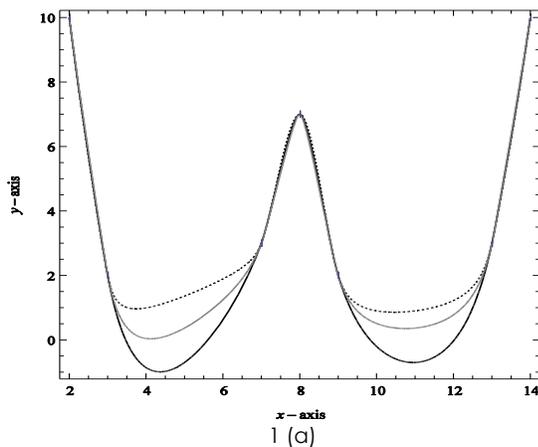
$$E_i = \alpha_i [\Delta_i(2\theta - 1) + (1-\theta)d_i - \theta d_{i+1}]$$

When  $\beta_i \rightarrow \infty$  or  $\alpha_i = 0$ , from (4) the rational cubic interpolant converges to straight line given below:

$$\lim_{\alpha_i, \beta_i \rightarrow \infty} S_i(x) = (1-\theta)f_i + \theta f_{i+1} \quad (5)$$

### 2.2 Shape Control of Rational Cubic Ball Interpolant

Figure 1 shows the examples of shape control of the rational cubic Ball interpolant defined in (1) by using data sets from Hussain and Sarfraz [9] given in Table 1.



**Figure 1** Shape control of rational cubic Ball interpolation for data in Table 1 with:

(a)  $\alpha_i = \beta_i = 1$  (black),  $\alpha_i = 1, \beta_i = 5$  (gray) and  $\alpha_i = 1, \beta_i = 20$  (dashed)

(b)  $\alpha_i = \beta_i = 1$  (black),  $\alpha_i = 5, \beta_i = 1$  (gray) and  $\alpha_i = 20, \beta_i = 1$  (dashed)

**Table 1** A data from Hussain and Sarfraz [9]

$i$	$x_i$	$f_i$	$d_i$
1	2	10	-9.65
2	3	2	-6.35
3	7	3	3.25
4	8	7	0
5	9	2	-3.95
6	13	3	5.65
7	14	10	8.35

### 2.3 Derivative Estimation

Since we are dealing with scalar data sets i.e. functional interpolation, thus the first derivative values need to be estimated by using some method. In this paper, the arithmetic mean method (AMM) are used. The formulation of AMM is further elaborated as follows:

At the end points  $x_1$  and  $x_n$

$$d_1 = \Delta_1 + (\Delta_1 - \Delta_2) \left( \frac{h_1}{h_1 + h_2} \right) \quad (6)$$

$$d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2}) \left( \frac{h_{n-1}}{h_{n-1} + h_{n-2}} \right) \quad (7)$$

At the interior points,  $x_i, i = 2, 3, \dots, n-1$ , the values of  $d_i$  are given as

$$d_i = \frac{h_{i-1}\Delta_i + h_i\Delta_{i-1}}{h_{i-1} + h_i} \quad (8)$$

### 2.4 Positivity Preserving Using Rational Cubic Ball Interpolant

The sufficient condition for the positivity of the rational cubic Ball interpolant,  $S_i(x)$  defined by (1) will be developed in this section. We assume that the strictly positive set of data  $\{(x_i, f_i), i = 1, 2, \dots, n\}$  are given, where

$$f_i > 0, \quad i = 1, 2, \dots, n \tag{9}$$

The rational cubic Ball interpolant  $S_i(x) > 0$  if and only if both  $P_i(\theta) > 0$  and  $Q_i(\theta) > 0$ . Clearly the denominator  $Q_i(\theta) > 0$  for all  $\alpha_i, \beta_i > 0, i = 1, 2, \dots, n - 1$ . Thus the positivity of the rational interpolant  $S(x)$  is depend to the positivity of the numerator  $P_i(\theta) > 0, i = 1, 2, \dots, n - 1$ . What we need to do is that find the sufficient conditions on shape parameters that satisfy  $P_i(\theta) > 0$ .

**Theorem 1** (Positivity of Cubic polynomial [22])

For the strict inequality positive data in (9),  $P_i(\theta) > 0$  if and only if

$$(P_i'(0), P_i'(1)) \in R_1 \cup R_2 \tag{10}$$

where

$$R_1 = \left\{ (a, b) : a > \frac{-3P_i(0)}{h_i}, b < \frac{3P_i(1)}{h_i} \right\}, \tag{11}$$

$$R_2 = \left\{ (a, b) : 36f_i f_{i+1} (a^2 + b^2 + ab - 3\Delta_i(a+b) + 3\Delta_i^2) + 3(f_{i+1}a - f_i b)(2h_i ab - 3f_{i+1}a + 3f_i b) + 4h_i (f_{i+1}a^3 - f_i b^3) - h_i^2 a^2 b^2 > 0 \right\} \tag{12}$$

with  $a = P_i'(0), b = P_i'(1)$  and  $P_i(0) = \alpha_i f_i, P_i(1) = \beta_i f_{i+1}$ .

For strictly positive data sets in (9), the following theorem gives the sufficient conditions for the positivity of the rational cubic Ball interpolant. It is data dependent and has one free parameter to alter the final positive interpolating curves.

**Theorem 2.** For a strictly positive data defined in (9), the rational cubic Ball interpolant defined on  $[x_1, x_n]$  is positive if in each subinterval  $[x_i, x_{i+1}], i = 1, 2, \dots, n - 1$  the following sufficient conditions are satisfied:

$$\alpha_i > 0,$$

$$\beta_i > \text{Max} \left\{ 0, -\frac{\alpha_i h_i d_i}{f_i}, \frac{\alpha_i h_i d_{i+1}}{f_{i+1}} \right\} \tag{13}$$

**Proof.**

By using (11), the following two inequalities will be obtained:

$$\frac{-2\alpha_i f_i + V_i}{h_i} > -\frac{3\alpha_i f_i}{h_i} \tag{14}$$

and

$$\frac{2\alpha_i f_{i+1} - W_i}{h_i} < \frac{3\alpha_i f_{i+1}}{h_i} \tag{15}$$

Simple algebraic manipulation to (14) and (15) gives the following conditions:

$$\beta_i f_i + \alpha_i h_i d_i > -3\alpha_i f_i \tag{16}$$

and

$$-\beta_i f_{i+1} + \alpha_i h_i d_{i+1} < 3\alpha_i f_{i+1} \tag{17}$$

For  $\alpha_i > 0$ , then the simplest conditions that will guarantee the positivity of the cubic polynomial  $P_i(\theta), i = 1, 2, \dots, n - 1$  is

$$\beta_i f_i > -\alpha_i h_i d_i \tag{18}$$

and

$$\beta_i f_{i+1} > \alpha_i h_i d_{i+1} \tag{19}$$

Thus combining (18) and (19) leads us to the following sufficient conditions for the positivity of rational cubic spline defined by (1):

$$\beta_i > \text{Max} \left\{ 0, -\frac{\alpha_i h_i d_i}{f_i}, \frac{\alpha_i h_i d_{i+1}}{f_{i+1}} \right\} \tag{20}$$

with  $\alpha_i > 0$ .

For the purpose of computer implementation, the above condition can be rewrite as:

$$\beta_i = \delta_i + \text{Max} \left\{ 0, -\frac{\alpha_i h_i d_i}{f_i}, \frac{\alpha_i h_i d_{i+1}}{f_{i+1}} \right\}, \quad \alpha_i, \delta_i > 0 \tag{21}$$

Algorithm for computer implementation:

**Input: Data points**

**Output: Parameters and positive interpolating curves**

Step 1: Input data points  $\{(x_i, f_i), i = 1, 2, \dots, n\}$

Step 2: For  $i = 1, \dots, n$

- Calculate the first derivative values by using AMM

Step 3: For  $i = 1, \dots, n - 1$

- Initialize free parameter  $\alpha_i > 0$
- Calculate the parameter values,  $\beta_i > 0$  by using the sufficient condition given in (21).

- Repeat by choosing different values of free parameter  $\alpha_i > 0$

Step 4: For  $i=1, \dots, n-1$

- Construct the positive interpolating curves with  $C^1$  continuity.

### 3.0 RESULTS AND DISCUSSION

In order to illustrate the shape preserving interpolation by using the proposed rational cubic Ball interpolation (cubic/quadratic), three positive data are taken from Tahat *et al.* [23], Jaafar *et al.* [13] and Abbas *et al.* [1] respectively.

**Table 2** data from Tahat *et al.* [23]

$i$	$x_i$	$f_i$	$d_i$
1	0	0.5	-2.27
2	0.25	0.1	-0.93
3	0.6	0.1	0.68
4	1.2	1.2	-0.32
5	1.6	0.5	-0.45
6	2.2	1.4	0.15
7	2.6	0.5	-1.13
8	3	0.5	-0.75
9	3.4	0.5	-0.22
10	3.9	0.25	-0.5
11	4	0.2	-0.5

**Table 3** data from Jaafar *et al.* [13]

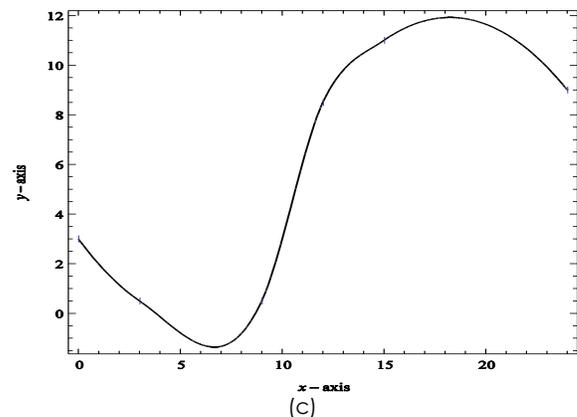
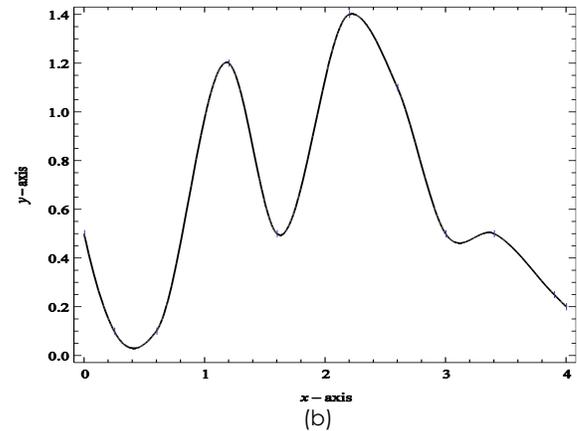
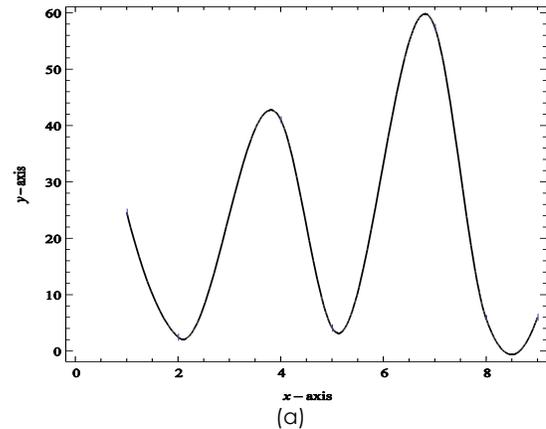
$i$	$x_i$	$f_i$	$d_i$
1	1	24.6162	-35.967
2	2	2.4616	-8.342
3	4	41.0270	-18.189
4	5	4.1027	15.727
5	7	57.4378	-25.574
6	8	5.7438	-25.719
7	9	6	26.231

**Table 4** data from Abbas *et al.* [1]

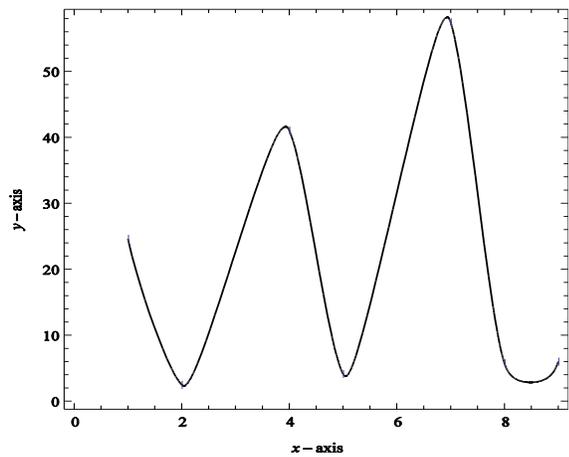
$i$	$x_i$	$f_i$	$d_i$
1	0	3	-1.11
2	3	0.5	-0.56
3	9	0.5	1.78
4	12	8.5	1.75
5	15	11	0.57
6	24	9	-1.01

For positive data in Table 3, it was noticed that the standard cubic Ball polynomial interpolation already preserves the shape of the data. In Jaafar *et al.* [13], the authors shows that the polynomial cubic Ball interpolation does not preserves the positivity of the data sets. From Figure 2(b) clearly our proposed

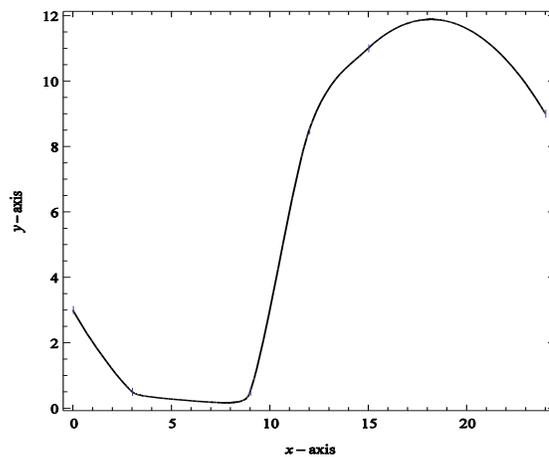
rational cubic Ball interpolant with default curves already preserves the positivity of the data sets. The interpolating curves also very smooth and comparable with the work of Jaafar *et al.* [13]. For data sets in Table 3, the user may use the proposed rational cubic Ball interpolant since with standard cubic Ball polynomial the positivity of the data are already preserved and the interpolating curves also very smooth even though with  $C^1$  continuity.



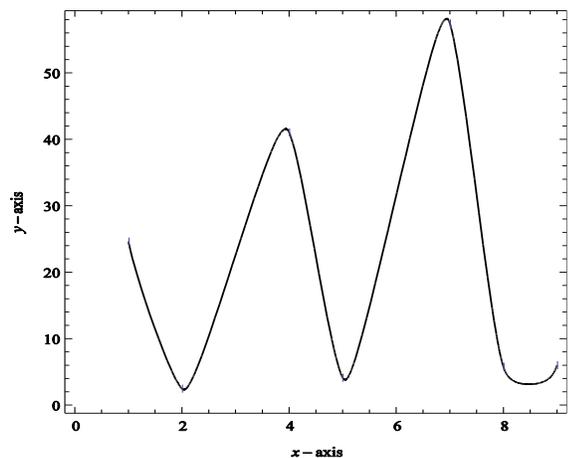
**Figure 2** Default cubic Ball interpolation with  $\alpha_i = 1, \beta_i = 0$  for data in: (a) Table 2 (b) Table 3 and (c) Table 4 respectively



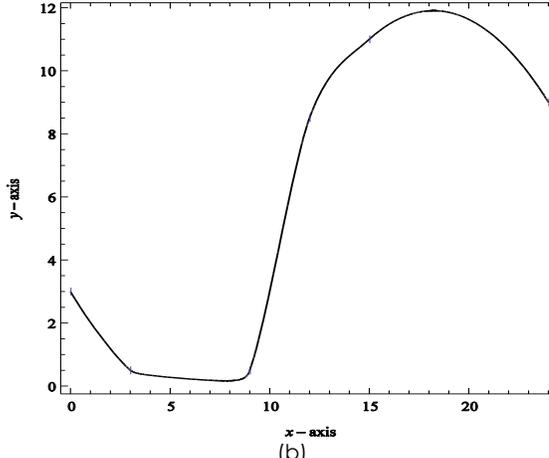
(a)



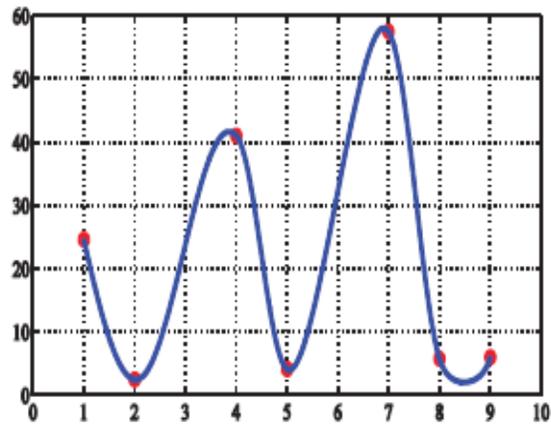
(a)



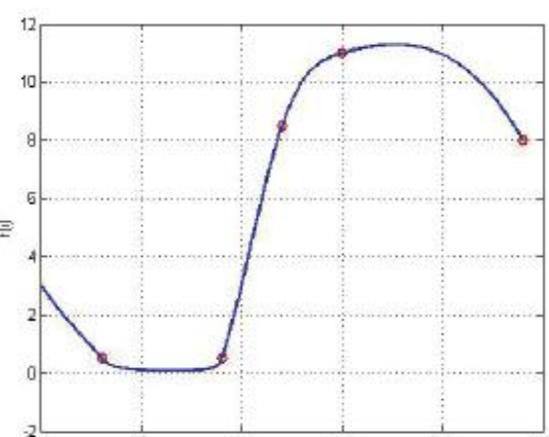
(b)



(b)



(c)

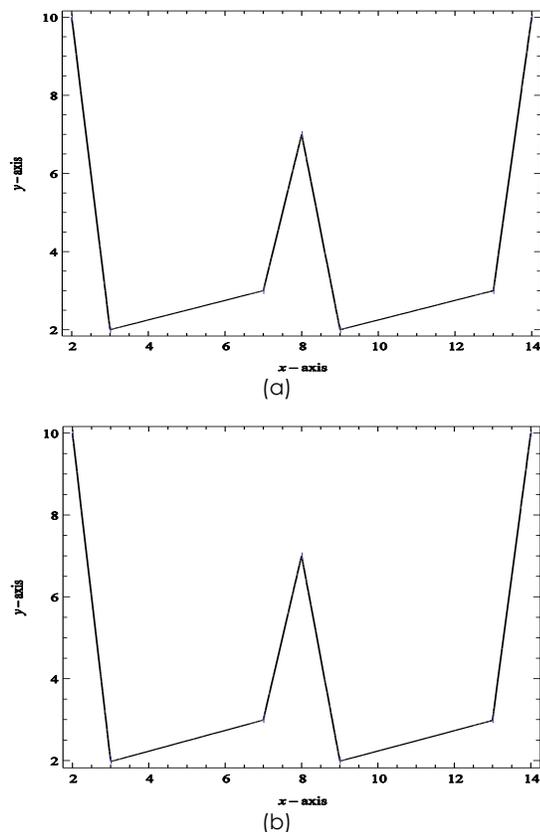


(c)

**Figure 3** Positivity preserving using the rational cubic Ball interpolation for data in Table 2 with: (a)  $\alpha_i = 1$  and (b)  $\alpha_i = 0.1$  and (c) the positivity using Tahat *et al.* [23]

**Figure 4** Positivity preserving using the rational cubic Ball interpolation for data in Table 4 with: (a)  $\alpha_i = 1$  and (b)  $\alpha_i = 2.5$  and (c) the schemes by Jaafar *et al.* [13]

Figure 3 and Figure 4 show the examples of positivity preserving by using the proposed schemes including comparison with existing schemes. Finally Figure 5 shows the examples that the rational cubic Ball interpolant defined by (1) converges to the straight line when  $\alpha_i \rightarrow 0$  or  $\beta_i \rightarrow \infty$  respectively.



**Figure 5** Linear reproducing for rational cubic Ball interpolant

## 4.0 CONCLUSION

In this study, new  $C^1$  rational cubic Ball spline with two parameters has been constructed. The sufficient conditions for positivity are derived on one parameter, meanwhile the other parameter is used to alter the resulting positive interpolating curves. Thus the proposed scheme has one degree of freedom. Comparison with the works of Tahat *et al.* [23] and Jaafar *et al.* [13] also have been done in details. From the results it can be seen clearly that the proposed rational cubic Ball spline works very well and give smooth results as well as visually pleasing and are better than both schemes presented in [13] and [23] respectively. Work on constrained data modeling, convexity and monotonicity preserving interpolation as well as 3D problems are underway. Furthermore constrained data interpolation ([24] and [6]) also very interesting topics and all the results will be reported soon.

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