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A MODIFIED DECOMPOSITION METHOD FOR SOLVING NONLINEAR PROBLEM OF FLOW IN CONVERGING-DIVERGING CHANNEL

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Abstract

In this research, an efficient technique of computation considered as a modified decomposition method was proposed and then successfully applied for solving the nonlinear problem of the two dimensional flow of an incompressible viscous fluid between nonparallel plane walls. In fact this method gives the nonlinear term *Nu* and the solution of the studied problem as a power series. The proposed iterative procedure gives on the one hand a computationally efficient formulation with an acceleration of convergence rate and on the other hand finds the solution without any discretization, linearization or restrictive assumptions. The comparison of our results with those of numerical treatment and other earlier works shows clearly the higher accuracy and efficiency of the used Modified Decomposition Method.

Keywords: Inclined walls, Jeffery-Hamel flow, Nonlinear problem, Velocity profiles, Skin friction, Modified Decomposition Method (MDM).

1. Introduction

The radial two dimensional flow of an incompressible viscous fluid between two inclined plane walls separated by an angle 2α driven by a line source or sink known as Jeffery-Hamel flow, is one of the few exact solutions of the Navier-Stokes equations. The nonlinear governing equation of this flow has been discovered by Jeffery [1] and independently by Hamel [2]. Since then, several contributions have been conducted in order to solve the nonlinear problem of Jeffery-Hamel flow. Indeed, Rosenhead [3] gives on the one hand the solution in terms of elliptic function

Nomenclatures

A_n	Adomian polynomials
a	Constant of divergent channel
b	Constant of divergent channel
c	Constant of convergent channel
c_f	Skin friction coefficient
d	Derivative operator
F	Non-dimensional velocity or general nonlinear operator
F'	First derivative of velocity
F''	Second derivative of velocity
f	Velocity
f_{max}	Velocity at the centerline of channel
g	Function
Lu	Linear operator
L^{-1}	Inverse or integral operator
n	Number of iteration
Nu	Nonlinear operator
P	Fluid pressure, N/m^2
r	Radial coordinate, m
Re	Reynolds number
R_{ec}	Critical Reynolds number
Ru	Remainder of linear operator
u	Function or velocity
V_{max}	Maximal velocity, m/s
V_r	Radial velocity, m/s
V_z	Axial velocity, m/s
V_θ	Azimuthal velocity, m/s
x	Cartesian coordinate, m
z	Axial coordinate, m

Greek Symbols

η	Non-dimensional angle
θ	Angular coordinate, deg.
α	Channel half-angle, deg.
α_c	Critical channel half-angle, deg.
φ	Constant
ρ	Density, kg/m^3
ν	Kinematic viscosity, m^2/s
∂	Derivative operator
∞	Condition at infinity

Abbreviations

ADM	Adomian decomposition method
HPM	Homotopy perturbation method
MDM	Modified decomposition method

and discuss on the other hand the various mathematically possible types of this flow. The exact solution of thermal distributions of the Jeffery-Hamel flow has been given by Millsaps and pohlhausen [4]. In this contribution, temperature distributions due only to dissipation have been calculated numerically in converging-diverging channels. Fraenkel [5] investigated the laminar flow in symmetrical channels with slightly curved walls. In this study the resulting velocity profiles of the flow were obtained as a power series. The nonlinear problem of the Jeffery-Hamel flow in converging-diverging channels has been also well discussed in many textbooks [6-8].

The two dimensional stability of Jeffery-Hamel flow has been extensively studied by many authors. Uribe et al. [9] used the Galerkin method to study linear and temporal stability of some flows for small widths of the channel. Sobey et al. [10] in their study suggested that the radial flows (convergent or divergent) are unstable. Banks et al. [11] treated the temporal and spatial stability of Jeffery-Hamel flow. Critical values of Reynolds numbers for Jeffery-Hamel flow have been computed by Hamadiche et al. [12]. Makinde et al. [13] also studied the temporal stability of the Jeffery-Hamel flow in convergent-divergent channels under the effect of an external magnetic field. The principal eigenvalues for some family of Jeffery-Hamel flows in diverging channels have been calculated by CARMI [14]. The linear temporal three dimensional stability of incompressible viscous flow in a rotating divergent channel is studied by Al Farkh et al. [15]. Eagles [16] for his part, investigated the linear stability of Jeffery-Hamel flows by solving numerically the relevant Orr-Sommerfeld problem.

In recent years several methods were developed in order to solve analytically the nonlinear initial or boundary value problem, such as the homotopy method [17-20], the variational iteration method [21-23] and the Adomian decomposition method [24-26]. These methods have been successfully applied for solving mathematical and physical problems. Indeed, many authors [27-31] have used these new approximate analytical methods to study the nonlinear problem of Jeffery-Hamel flow. Motsa et al. for their part [32] undertook a study on the nonlinear problem of magnetohydrodynamic Jeffery-Hamel flow by using a novel hybrid spectral homotopy analysis technique. The Adomian decomposition method has been also successfully used [33, 34] in the solution of Blasius equation and the two dimensional laminar boundary layer of Falkner-Skan for wedge.

In the present research, the solution of two dimensional flow of an incompressible viscous fluid between nonparallel plane walls is presented. The resulting third order differential equation is solved analytically by a modified decomposition method and the obtained results are compared to the numerical results. The principal aim of this study is on the one hand to find approximate analytical solutions of Jeffery-Hamel flow in convergent-divergent channels and on the other hand to investigate the accuracy and efficiency of the adopted Modified Decomposition Method.

2. Governing Equations

In this research the flow of an incompressible viscous fluid between nonparallel plane walls has been studied. The geometrical configuration of the Jeffery-Hamel flow is given in Fig. 1. Indeed, the considered flow is uniform along the *z*-direction and we assume purely radial motion, i.e.: for velocity components we can write: $(V_r = V(r,\theta); V_\theta = V_z = 0)$

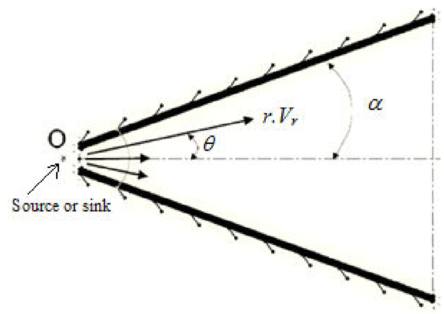


Fig. 1. Geometry of Jeffery-Hamel flow.

In cylindrical coordinates (r, θ, z) the reduced forms of continuity and Navier-Stokes equations, are:

$$\frac{\rho}{r}\frac{\partial}{\partial r}\left(rV_r\right)=0 \tag{1}$$

$$V_r\frac{\partial V_r}{\partial r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\nu\left[\frac{\partial^2 V_r}{\partial r^2}+\frac{1}{r}\frac{\partial V_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}-\frac{V_r}{r^2}\right] \tag{2}$$

$$-\frac{1}{\rho.\mathrm{r}}\frac{\partial P}{\partial \theta}+\frac{2\nu}{\mathrm{r}^2}\frac{\partial V_r}{\partial \theta}=0 \tag{3}$$

where V_r : radial velocity; ρ : density ; ν : kinematic viscosity ; P : fluid pressure.

According to Eq. (1), we define that the quantities (r,V_r) depends on θ and we can write:

$$rV_r=f(\theta) \tag{4}$$

By considering (4) and eliminating the pressure term between Eqs. (2) and (3), we obtain:

$$f^{'''}+4f^{'}+\frac{2}{\nu}ff^{''}=0 \tag{5}$$

If we introduce the dimensionless parameter $F(\eta)=\frac{f(\theta)}{f_{\max}}$, where: $\eta=\frac{\theta}{\alpha}$ with: $-1\le\eta\le+1$.

And considering the following quantities obtained after normalization:

$$f^{'}(\theta)=\frac{1}{\alpha}F^{'}(\eta)\,,\ f^{''}(\theta)=\frac{1}{\alpha^2}F^{''}(\eta)\,,\ f^{'''}(\theta)=\frac{1}{\alpha^3}F^{'''}(\eta) \tag{6}$$

Finally the Eq. (5) in non-dimensional form can be reduced to an ordinary differential equation:

$$F^{'''}(\eta)+2R_e\alpha F(\eta)F^{'}(\eta)+4\alpha^2F^{'}(\eta)=0 \tag{7}$$

The Reynolds number of flow is introduced as:

$$R_e=\frac{rV_{\max}\alpha}{\nu}=\frac{f_{\max}\alpha}{\nu} \tag{8}$$

where f_{max} is the velocity at the centerline of channel, and α is the channel half-angle.

The boundary conditions of the Jeffery-Hamel flow in terms of $F(\eta)$ are expressed as follows:

at the centerline of channel $F(\eta)=1, F^{'}(\eta)=0$ (9)

at the body of channel $F(\pm\eta)=0$ (10)

3. Modified Decomposition Method Formulation

Consider the differential equation:

$$Lu + Ru + Nu = g(t) \quad (11)$$

where: N is a nonlinear operator, L is the highest ordered derivative and R represents the remainder of linear operator L.

By considering L^{-1} as an n-fold integration for an nth order of L, the principles of method consists on applying the operator L^{-1} to the expression (11). Indeed, we obtain:

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \quad (12)$$

The solution of Eq. (12) is given by:

$$u = \varphi + L^{-1}g - L^{-1}Ru - L^{-1}Nu \quad (13)$$

where φ is determined from the boundary or initial conditions.

Normally for the standard Adomian decomposition method, the solution u can be determined as an infinite series with the components given by:

$$u = \sum_{n=0}^{\infty} u_n \quad (14)$$

And the nonlinear term Nu is given as following:

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots\ldots., u_n) \quad (15)$$

where $A_n's$ is called Adomian polynomials and has been introduced by George Adomian [24] by the recursive formula:

$$A_n(u_0, u_1, \ldots\ldots., u_n) = \frac{1}{n\,!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\right]_{\lambda=0}, n = 0,1,2,\ldots\ldots., n \quad (16)$$

By substituting the given series (14), (15) into both sides of (13), we obtain the following expressions:

$$\sum_{n=0}^{\infty} u_n = \varphi + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n \quad (17)$$

According to Eq. (17), the recursive expression which defines the ADM components u_n is given as:

$$u_0 = \varphi + L^{-1}g, u_{n+1} = -L^{-1}(Ru_n + A_n) \quad (18)$$

For the adopted modified decomposition method, based on the idea of power series method, we assume that the solution u and the nonlinear term Nu can be decomposed respectively as:

$$u = \sum_{n=0}^{\infty} u_n x^n \quad (19)$$

$$u = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots\ldots., u_n)x^n \quad (20)$$

When Adomian polynomials A_n are known, the components of the solution u, for the adopted Modified Decomposition Method are expressed as follows:

$$u_0 = \varphi + L^{-1}g, u_{n+1} = -L^{-1}(Ru_n + A_n)x^n \tag{21}$$

Finally after some iteration, the solution of the studied equation can be given as a power series as follows:

$$u = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + ... + +u_n x^n \tag{22}$$

4. Application of MDM to the Jeffery-Hamel Problem

Considering the Eq. (11), Eq. (7) can be written as:

$$LF = -2R_e \alpha FF' - 4\alpha^2 F' \tag{23}$$

where the differential operator L is given by: $L = \frac{d^3}{dn^3}$.

The inverse of operator L is expressed by L^{-1} and can be represented as:

$$L^{-1} = \int_0^\eta \int_0^\eta \int_0^\eta (\bullet) d\eta d\eta d\eta \tag{24}$$

The application of Eq. (24) on Eq. (23) and considering the boundary conditions (9) and (10), we obtain:

$$F(\eta) = F(0) + F'(0)\eta + F''(0)\frac{\eta^2}{2} + L^{-1}(Nu) \tag{25}$$

where:

$$Nu = -2R_e \alpha FF' - 4\alpha^2 F' \tag{26}$$

The values of $F(0), F'(0)$ and $F''(0)$ depend on boundary conditions for convergent-divergent channels. In order to distinguish between convergent and divergent channels, the boundary conditions are taken with different manner [29]. Indeed, on the one hand for divergent channel, we assume that the body of channel is given by $\eta = 0$ and by considering a symmetric condition in the center of channel; the solution is studied between $F(0) = 0$ at the body of channel and $F(1) = 1$ at the centerline of channel. On the other hand, for convergent channels, the center of channel is given by $\eta = 0$ and consequently the solution varies from $F(1) = 0$ at the body of channel to $F(0) = 1$ at the centerline of channel.

4.1. Convergent channel:

For convergent channel the boundary conditions are expressed as follows:

at the centerline of channel $F(0) = 1, F'(0) = 0$ (27)

at the body of channel $F(1) = 0$ (28)

In this case, the solution of our problems is studied between $F(1)=0$ at the body of channel and $F(0)=1$ at the centerline of channel. By applying the boundary conditions (27), (28) and considering $F''(0)=c$, we obtain:

$$F(\eta)=\sum_{n=0}^{\infty} F_n = F_0 + L^{-1}(Nu) \tag{29}$$

where:

$$F_0 = 1 + c\frac{\eta^2}{2} \tag{30}$$

According to the modified decomposition method, Adomian polynomials and solutions terms will be computed as follows:

$$A_0 = -2cR_e\alpha\eta - 4c\alpha^2\eta - c^2R_e\alpha\eta^3 \tag{31}$$

$$F_1 = -\frac{1}{12}cR_e\alpha\eta^4 - \frac{1}{6}c\alpha^2\eta^4 - \frac{1}{120}c^2R_e\alpha\eta^6 \tag{32}$$

$$\begin{aligned} A_1 &= +\frac{1}{6}c^2R_e{}^2\alpha^2\eta^4 + \frac{2}{3}c^2R_e\alpha^3\eta^4 + \frac{2}{3}c^2\alpha^4\eta^4 \\ &+ \frac{4}{15}c^3R_e{}^2\alpha^2\eta^6 + \frac{8}{15}c^3R_e\alpha^3\eta^6 + \frac{1}{40}c^4R_e{}^2\alpha^2\eta^8 \end{aligned} \tag{33}$$

$$\begin{aligned} F_2 &= +\frac{1}{2016}c^2R_e{}^2\alpha^2\eta^8 + \frac{1}{504}c^2R_e\alpha^3\eta^8 + \frac{1}{504}c^2\alpha^4\eta^8 \\ &+ \frac{1}{2700}c^3R_e{}^2\alpha^2\eta^{10} + \frac{1}{1350}c^3R_e\alpha^3\eta^{10} + \frac{1}{52800}c^4R_e{}^2\alpha^2\eta^{12} \end{aligned} \tag{34}$$

$$\begin{aligned} A_2 &= -\frac{1}{1008}c^3R_e{}^3\alpha^3\eta^8 - \frac{1}{168}c^3R_e{}^2\alpha^4\eta^8 - \frac{1}{84}c^3R_e\alpha^5\eta^8 - \frac{1}{126}c^3\alpha^6\eta^8 \\ &- \frac{1}{48}c^4R_e{}^3\alpha^3\eta^9 - \frac{1}{12}c^4R_e{}^2\alpha^4\eta^9 - \frac{1}{12}c^4R_e\alpha^5\eta^9 - \frac{337}{151200}c^4R_e{}^3\alpha^3\eta^{10} \\ &- \frac{337}{37800}c^4R_e{}^2\alpha^4\eta^{10} - \frac{337}{37800}c^4R_e\alpha^5\eta^{10} - \frac{1}{240}c^5R_e{}^3\alpha^3\eta^{11} \\ &- \frac{1}{120}c^5R_e{}^2\alpha^4\eta^{11} - \frac{91}{79200}c^5R_e{}^3\alpha^3\eta^{12} - \frac{91}{39600}c^5R_e{}^2\alpha^4\eta^{12} \\ &- \frac{1}{4800}c^6R_e{}^3\alpha^3\eta^{13} - \frac{1}{17600}c^6R_e{}^3\alpha^3\eta^{14} \end{aligned} \tag{35}$$

$$\begin{aligned} F_3 &= -\frac{1}{1729728}c^3R_e{}^3\alpha^3\eta^{13} - \frac{1}{288288}c^3R_e{}^2\alpha^4\eta^{13} - \frac{1}{144144}c^3R_e\alpha^5\eta^{13} \\ &- \frac{1}{216216}c^3\alpha^6\eta^{13} - \frac{1}{104832}c^4R_e{}^3\alpha^3\eta^{14} - \frac{1}{26208}c^4R_e{}^2\alpha^4\eta^{14} \\ &- \frac{1}{26208}c^4R_e\alpha^5\eta^{14} - \frac{337}{412776000}c^4R_e{}^3\alpha^3\eta^{15} - \frac{337}{103194000}c^4R_e{}^2\alpha^4\eta^{15} \\ &- \frac{337}{103194000}c^4R_e\alpha^5\eta^{15} - \frac{1}{806400}c^5R_e{}^3\alpha^3\eta^{16} - \frac{1}{403200}c^5R_e{}^2\alpha^4\eta^{16} \\ &- \frac{91}{323136000}c^5R_e{}^3\alpha^3\eta^{17} - \frac{91}{161568000}c^5R_e{}^2\alpha^4\eta^{17} \\ &- \frac{1}{23500800}c^6R_e{}^3\alpha^3\eta^{18} - \frac{1}{102326400}c^6R_e{}^3\alpha^3\eta^{19} \end{aligned} \tag{36}$$

Finally, the solution for convergent channel is given by the modified decomposition method as a power series as following:

$$F(\eta)=F_0+F_1\eta+F_2\eta^2+F_3\eta^3+........+F_n\eta^n \quad (37)$$

The value of constant c is obtained by solving Eq. (37) using Eq. (28).

4.2. Divergent channel

For divergent channel, the boundary conditions are expressed as following:

In the body of channel: $F(0)=0$ (38)

At the centerline of channel: $F(1)=1$. (39)

In this case, the solution varies from $F(0)=0$ at the body of channel to $F(1)=1$ at the centerline of channel.

According to the boundary conditions (38), (39) and assuming that $a=F^{'}(0)$ and $b=F^{''}(0)$, the solution is finally given as follows:

$$F(\eta)=\sum_{n=0}^{\infty}F_n=F_0+L^{-1}(Nu) \quad (40)$$

where: $F_0=a\eta+b\dfrac{\eta^2}{2}$ (41)

By applying the modified decomposition algorithm, the terms of solution and Adomian polynomials for divergent channel are expressed as follows:

$$A_0=-4a\alpha^2-2a^2R_e\alpha\eta-4b\alpha^2\eta-3abR_e\alpha\eta^2-b^2R_e\alpha\eta^3 \quad (42)$$

$$F_1=-\frac{2}{3}a\alpha^2\eta^3-\frac{1}{12}a^2R_e\alpha\eta^4-\frac{1}{6}b\alpha^2\eta^4-\frac{1}{20}abR_e\alpha\eta^5-\frac{1}{120}b^2R_e\alpha\eta^6 \quad (43)$$

$$\begin{aligned}A_1&=\frac{4}{3}a^3R_e\alpha^3\eta^3+\frac{8}{3}ab\alpha^4\eta^3+\frac{1}{6}a^4R_e^{\,2}\alpha^2\eta^4+\frac{14}{3}a^2bR_e\alpha^3\eta^4\\&+\frac{2}{3}b^2\alpha^4\eta^4+\frac{3}{5}a^3bR_e^{\,2}\alpha^2\eta^5+\frac{16}{5}ab^2R_e\alpha^3\eta^5+\frac{17}{30}a^2b^2R_e^{\,2}\alpha^2\eta^6\\&+\frac{8}{15}b^3R_e\alpha^3\eta^6+\frac{1}{5}ab^3R_e^{\,2}\alpha^2\eta^7+\frac{1}{40}b^4R_e^{\,2}\alpha^2\eta^8\end{aligned} \quad (44)$$

$$\begin{aligned}F_2&=\frac{2}{315}a^3R_e\alpha^3\eta^7+\frac{4}{315}ab\alpha^4\eta^7+\frac{1}{2016}a^4R_e^{\,2}\alpha^2\eta^8+\frac{1}{72}a^2bR_e\alpha^3\eta^8\\&+\frac{1}{504}b^2\alpha^4\eta^8+\frac{1}{840}a^3bR_e^{\,2}\alpha^2\eta^9+\frac{2}{315}ab^2R_e\alpha^3\eta^9+\frac{17}{21600}a^2b^2R_e^{\,2}\alpha^2\eta^{10}\\&+\frac{1}{1350}b^3R_e\alpha^3\eta^{10}+\frac{1}{4950}ab^3R_e^{\,2}\alpha^2\eta^{11}+\frac{1}{52800}b^4R_e^{\,2}\alpha^2\eta^{12}\end{aligned} \quad (45)$$

$$
\begin{aligned}
A_2 = &-\frac{4a^3bR_e\alpha^5\eta^6}{3}-\frac{4a^5R_e^2\alpha^4\eta^7}{315}-\frac{a^4bR_e^2\alpha^4\eta^7}{3}-\frac{16a^3bR_e\alpha^5\eta^7}{315}\\
&-2a^2b^2R_e\alpha^5\eta^7-\frac{16ab^2\alpha^6\eta^7}{315}-\frac{a^6R_e^3\alpha^3\eta^8}{1008}-\frac{a^5bR_e^3\alpha^3\eta^8}{48}-\frac{19a^4bR_e^2\alpha^4\eta^8}{280}\\
&-\frac{37a^3b^2R_e^2\alpha^4\eta^8}{60}-\frac{19a^2b^2R_e\alpha^5\eta^8}{140}-\frac{3ab^3R_e\alpha^5\eta^8}{4}-\frac{b^3\alpha^6\eta^8}{126}-\frac{3a^5bR_e^3\alpha^3\eta^9}{560}\\
&-\frac{11a^4b^2R_e^3\alpha^3\eta^9}{240}-\frac{151a^3b^2R_e^2\alpha^4\eta^9}{1260}-\frac{11a^2b^3R_e^2\alpha^4\eta^9}{30}-\frac{19ab^3R_e\alpha^5\eta^9}{252}\\
&-\frac{b^4R_e\alpha^5\eta^9}{12}-\frac{1543a^4b^2R_e^3\alpha^3\eta^{10}}{151200}-\frac{11a^3b^3R_e^3\alpha^3\eta^{10}}{300}-\frac{319a^2b^3R_e^2\alpha^4\eta^{10}}{3780}\\
&-\frac{11ab^4R_e^2\alpha^4\eta^{10}}{120}-\frac{337b^4R_e\alpha^5\eta^{10}}{37800}-\frac{2411a^3b^3R_e^3\alpha^3\eta^{11}}{277200}-\frac{17a^2b^4R_e^3\alpha^3\eta^{11}}{1200}\\
&-\frac{421ab^4R_e^2\alpha^4\eta^{11}}{17325}-\frac{b^5R_e^2\alpha^4\eta^{11}}{120}-\frac{13a^2b^4R_e^3\alpha^3\eta^{12}}{3600}-\frac{13ab^5R_e^3\alpha^3\eta^{12}}{4800}\\
&-\frac{91b^5R_e^2\alpha^4\eta^{12}}{39600}-\frac{19ab^5R_e^3\alpha^3\eta^{13}}{26400}-\frac{b^6R_e^3\alpha^3\eta^{13}}{4800}-\frac{b^6R_e^3\alpha^3\eta^{14}}{17600}
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
F_3 = &-\frac{2a^3bR_e\alpha^5\eta^{11}}{1485}-\frac{a^5R_e^2\alpha^4\eta^{12}}{103950}-\frac{a^4bR_e^2\alpha^4\eta^{12}}{3960}-\frac{2a^3bR_e\alpha^5\eta^{12}}{51975}-\frac{a^2b^2R_e\alpha^5\eta^{12}}{660}\\
&-\frac{2ab^2\alpha^6\eta^{12}}{51975}-\frac{a^6R_e^3\alpha^3\eta^{13}}{1729728}-\frac{a^5bR_e^3\alpha^3\eta^{13}}{82368}-\frac{19a^4bR_e^2\alpha^4\eta^{13}}{480480}-\frac{37a^3b^2R_e^2\alpha^4\eta^{13}}{102960}\\
&-\frac{19a^2b^2R_e\alpha^5\eta^{13}}{240240}-\frac{ab^3R_e\alpha^5\eta^{13}}{2288}-\frac{b^3\alpha^6\eta^{13}}{216216}-\frac{a^5bR_e^3\alpha^3\eta^{14}}{407680}-\frac{11a^4b^2R_e^3\alpha^3\eta^{14}}{524160}\\
&-\frac{151a^3b^2R_e^2\alpha^4\eta^{14}}{2751840}-\frac{11a^2b^3R_e^2\alpha^4\eta^{14}}{65520}-\frac{19ab^3R_e\alpha^5\eta^{14}}{550368}-\frac{b^4R_e\alpha^5\eta^{14}}{26208}-\frac{1543a^4b^2R_e^3\alpha^3\eta^{15}}{412776000}\\
&-\frac{11a^3b^3R_e^3\alpha^3\eta^{15}}{819000}-\frac{319a^2b^3R_e^2\alpha^4\eta^{15}}{10319400}-\frac{11ab^4R_e^2\alpha^4\eta^{15}}{327600}-\frac{337b^4R_e\alpha^5\eta^{15}}{103194000}\\
&-\frac{2411a^3b^3R_e^3\alpha^3\eta^{16}}{931392000}-\frac{17a^2b^4R_e^3\alpha^3\eta^{16}}{4032000}-\frac{421ab^4R_e^2\alpha^4\eta^{16}}{58212000}-\frac{b^5R_e^2\alpha^4\eta^{16}}{403200}-\frac{13a^2b^4R_e^3\alpha^3\eta^{17}}{14688000}\\
&-\frac{13ab^5R_e^3\alpha^3\eta^{17}}{19584000}-\frac{91b^5R_e^2\alpha^4\eta^{17}}{161568000}-\frac{19ab^5R_e^3\alpha^3\eta^{18}}{129254400}-\frac{b^6R_e^3\alpha^3\eta^{18}}{23500800}-\frac{b^6R_e^3\alpha^3\eta^{19}}{102326400}
\end{aligned}
\tag{47}
$$

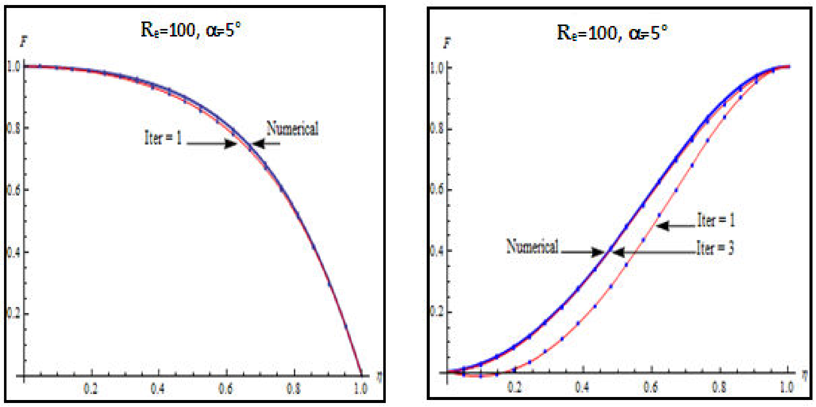
Finally, the solution for divergent channel is given by the modified decomposition method as a power series as following:

$$F(\eta) = F_0 + F_1\eta^{-1} + F_2\eta^{-2} + F_3\eta^{-3} + ... + F_n\eta^{-n} \tag{48}$$

The quantities a and b can be determined by solving obtained Eq. (48) using Eqs. (38) and (39).

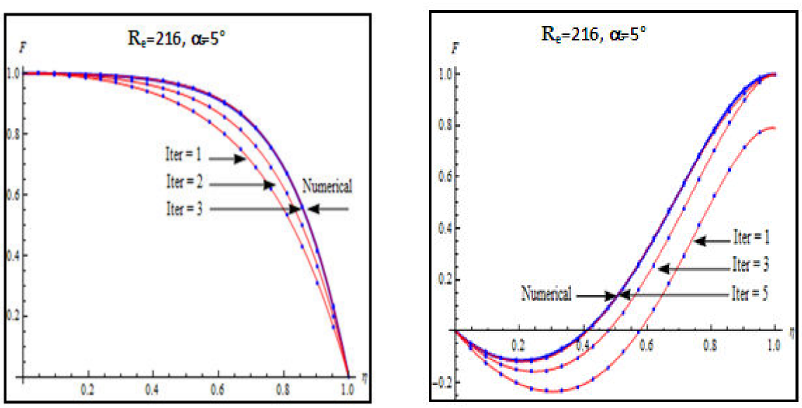
5. Results and discussions

In this study we are particularly interested in the nonlinear problem of the Jeffery-Hamel flow. The nonlinear differential equations (7) with boundary conditions (9) and (10) have been treated analytically for some values of the governing parameters Re and α using the modified decomposition method. Generally, the solution obtained by the modified decomposition method converges rapidly. According to Fig. 2, we notice that the solution in converging-diverging channels for small Reynolds number (*Re*= 100) converges quickly to the numerical solution. Indeed, analytical solution converges at the first iteration in convergent flow and in the third iteration for divergent flow. As it is clear in Fig. 3 when Reynolds number becomes higher (*Re*= 216), the obtained solution by modified decomposition method converges after 3 iterations for convergent channel and after 5 iterations for divergent channel.



a) Convergent Channel **b) Divergent Channel**

Fig. 2. Velocity profiles by MDM after iteration for Re=100, α=5°.



a) Convergent Channel **b) Divergent Channel**

Fig. 3. Velocity profiles by MDM after iteration for Re=216, α=5°.

In order to test the accuracy, applicability and efficiency of this new Modified Decomposition Method, a comparison with the numerical results obtained by fourth order Runge Kutta method is performed. We notice that the comparison shows an excellent agreement between analytical and numerical data for convergent and divergent channels, as presented in Tables 1 and 2. In these tables, the error is introduced as:

$$Error = \left|F(\eta)_{MDM} - F(\eta)_{Num}\right|'$$

As it is shown in Tables 3 and 4 in comparison with the standard Adomian method and the Homotopy perturbation method for diverging channel, we notice that the MDM technique has a high precision than ADM and HPM. Clearly, results show that the MDM solution converges quickly after five iteration of computation. It is also noted that both analytical and numerical results are in excellent agreement.

Table 1. Comparison between the Numerical and MDM results for velocity distribution in converging channel when: α=3°.

η	Re=100			Re=150			Re=300			Re=400		
	Numerical	MDM	Error	Numerical	MDM	Error	Numerical	MDM	Error	Numerical	MDM	Error
0	1.00000	1.00000	0.000000	1.000000	1.000000	0.000000	1.000000	1.000000	0.000000	1.000000	1.000000	0.000000
0.2	0.979311	0.979311	0.000000	0.984968	0.984968	0.000000	0.993841	0.993840	0.000001	0.996415	0.996413	0.000002
0.4	0.908482	0.908482	0.000000	0.930100	0.930100	0.000000	0.966850	0.966846	0.000004	0.978803	0.978790	0.000013
0.6	0.759108	0.759108	0.000000	0.801249	0.801249	0.000000	0.882613	0.882600	0.000013	0.914056	0.914003	0.000053
0.8	0.479108	0.479269	0.000161	0.527538	0.527538	0.000000	0.636771	0.636739	0.000032	0.688627	0.688475	0.000152
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 2 the comparison between the Numerical and MDM results for velocity in diverging channel when: α=3°.

η	Re=100			Re=150			Re=300			Re=400		
	Numerical	MDM	Error	Numerical	MDM	Error	Numerical	MDM	Error	Numerical	MDM	Error
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.213026	0.213019	0.000007	0.136611	0.136587	0.000024	-0.057253	-0.057280	0.000027	-0.145084	-0.145119	0.000035
0.4	0.466647	0.466635	0.000012	0.365996	0.365954	0.000040	0.081612	0.081558	0.000054	-0.059531	-0.059607	0.000076
0.6	0.724435	0.724424	0.000011	0.649619	0.649575	0.000056	0.406022	0.405950	0.000072	0.262916	0.262799	0.000117
0.8	0.923955	0.923949	0.000006	0.898668	0.898642	0.000026	0.805209	0.805157	0.000052	0.740270	0.740169	0.000101
1	1.000000	1.00000	0.000000	1.000000	1.000000	0.000000	1.000000	1.000000	0.000000	1.000000	1.000000	0.000000

Table 3 Comparison of the MDM results against the numerical and ADM for *F(η)* when *α*=5°, Re=50.

η	HPM [30]	Numerical	MDM (Present study)		ADM (Present study)		
	F(η)	*F(η)*	*F(η)*	*F(η)*	*F(η)*	*F(η)*	*F(η)*
			3rd order	**5th order**	**3rd order**	**5th order**	**9th order**
0	1	1	1	1	1	1	1
0.25	0.894960	0.894242	0.894342	0.894238	0.880756	0.891679	0.891602
0.50	0.627220	0.626948	0.627103	0.626942	0.604680	0.621332	0.619762
0.75	0.302001	0.301990	0.302102	0.301986	0.285037	0.296600	0.294271
1	0	0	0	0	0	0	0

Table 4 Comparison of the MDM results against the numerical and ADM for *F″(η)* when *α*=5°, Re=50.

η	HPM [30]	Numerical	MDM (Present study)		ADM (Present study)		
	F"(*η*)	F"(*η*)	F"(*η*)	F"(*η*)	F"(*η*)	F"(*η*)	F"(*η*)
			3rd order	5th order	3rd order	5th order	9th order
0	-3.539214	-3.539416	-3.530365	-3.539803	-4.637850	-3.712261	-3.63750
0.25	-2.661930	-2.662084	-2.663422	-2.662031	-2.513760	-2.665958	-2.69556
0.50	-0.879711	-0.879794	-0.881423	-0.879736	-0.643805	-0.824410	-0.812129
0.75	0.447331	0.447244	0.446170	0.447281	0.630888	0.530321	0.580548
1	0.854544	0.854369	0.853594	0.854395	1.011719	0.959673	1.0395

On the other hand as displayed in Figs. 4 and 5, it is also clearly demonstrated that the proposed MDM led to more appropriate results when compared with those of ADM and HPM. In fact, the results show that the modified decomposition method provides better approximations to the solution of nonlinear problem of Jeffery-Hamel flow with high accuracy.

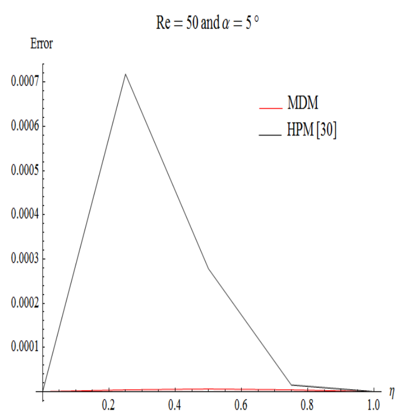


Fig. 4. The comparison between error of MDM and HPM solutions for *F(η)*, *α*=5° and Re=50.

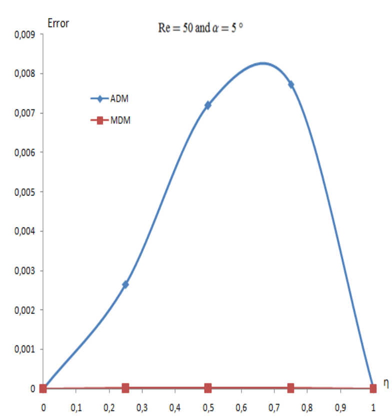


Fig. 5. The comparison between error of MDM and ADM solutions for *F(η)*, *α*=5° and Re=50.

With intention to show the importance of the studied flow, the numerical and analytical values are plotted in Figs. 6-15. Indeed, these figures show the velocity profiles in convergent-divergent channels.

Figure 6 shows the velocity profiles in channel of half angle, α = 7° and fixed Reynolds number (R_e= 126) for purely convergent flow. As it clear, the dimensionless velocity decreased from 1 at $\eta = 0$ to values 0 at $\eta = \pm 1$. The effect of Reynolds number on velocity profiles for convergent flow is depicted in Fig. 7. Indeed, increasing Reynolds number leads on the one hand to a flatter profile at the centre of channel with high gradients near the walls and on the other hand to decreased thickness of the boundary layer. We notice also that the velocity profiles are symmetric against $\eta = 0$ and the symmetric convergent flow is possible for opening angle 2α not exceed π. It is also clear for convergent channels that the backflow is excluded.

The variation of velocity profiles in divergent channels for a fixed Reynolds number (R_e= 126) when α = 7° is investigated in Fig. 8. Obtained results show that the velocity increased from 0 at $\eta = 0$ to 1 at $\eta = \pm 1$. The effect of Reynolds number on divergent flow (Fig. 9) is to concentrate the volume flux at the centre of channels with smaller gradients near the walls and consequently the thickness of boundary layer increase. For purely divergent channels, symmetric flow is not possible for an opening angle 2α, unless for Reynolds numbers not exceed a critical value. Above this critical Reynolds number, we observe clearly that the separation and backflow are started (see Fig. 10 and Table 5). In fact, the negative values of skin friction indicate on the beginning of the backflow phenomenon.

In Table 5 are listed the values of $F'(0), F''(0)$ and c_f. Indeed, $F'(0)$ and $F''(0)$ are investigated analytically by the modified decomposition method, but c_f is evaluated numerically by fourth order Runge Kutta method.

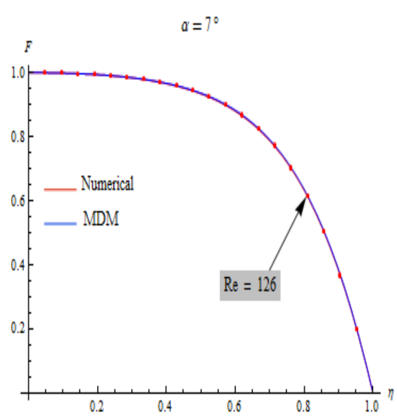


Fig. 6. Velocity profiles in convergent channels (Re=126 and α=7°).

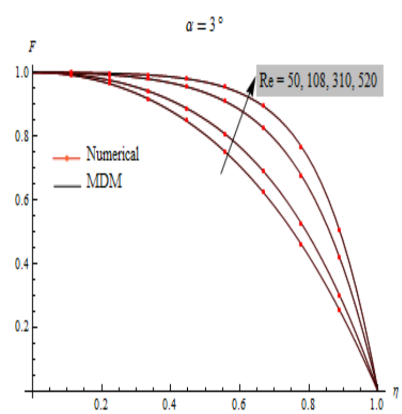


Fig. 7. Effect of Reynolds number on velocity profiles for convergent channels (α=3°).

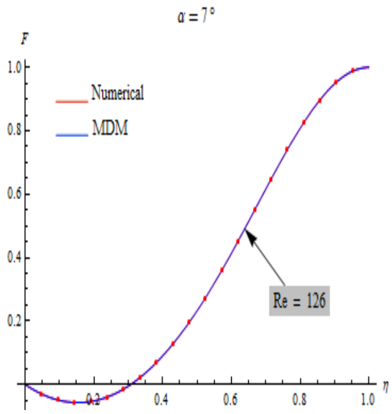


Fig. 8. Velocity profiles in divergent channels (Re=126 and α=7°).

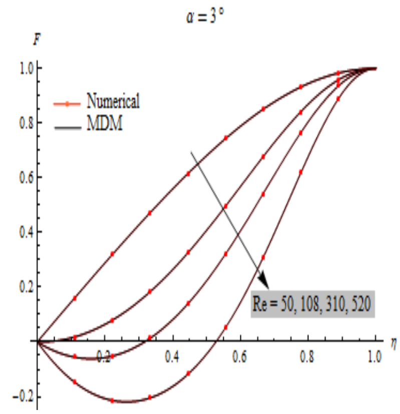


Fig. 9. Effect of Reynolds number on velocity profiles for divergent channels (α =3°).

It is also worth noting that the comparison of skin friction values shows a good agreement between analytical and numerical data. On the other hand as presented in Tables 6 and 7 for critical values of Reynolds number and channel half angle; results show a better agreement between MDM and numerical data. According to the obtained results, we notice that the magnitude of R_{ec} decreases with an increase in the channel half-angle α as shown in Table 6. We observe also in Table 7 that the critical channel half angle α_c decreases with increasing of Reynolds number. It is also interesting to note that above the critical values R_{ec} and α_c , the backflow phenomenon is started.

Table 5 Constant of divergent channel when α=3° (Analytical and numerical results).

		Diverging Channel (α = 3°)						
	Re	**100**	**150**	**200**	**250**	**300**	**400**	**500**
	a	0.938687	0.434542	-0.0252077	-0.429138	-0.775068	-1.31939	-1.72324
MDM	b	1.30992	2.52784	3.4938	4.27445	4.93803	6.10848	7.23028
Numerical	C_f	0.9376888	0.4318679	-0.02842059	-0.431812	-0.777680	-1.323495	-1.729066

Table 6 Computation showing the divergent channel flow critical Reynolds number.

α	3°	6°	9°	12°	15°
R_{ec} (Numerical)	198.000081	100.000042	66.000027	50.000021	40.000017
R_{ec} (MDM)	198.000080	100.000038	66.000029	50.000027	40.000016

Table 7. Computation showing the divergent channel flow critical half-angle of the channel, α_c.

Re	50	100	200	250	300
α_c ° (Numerical)	12.0000051	6.0000025	3.0000012	2.40642273	2.00535228
α_c ° (MDM)	12.0000056	6.0000024	3.0000015	2.40642271	2.00535225

In Fig. 11 we compare critical Reynolds numbers based on axial velocity, obtained by MDM and fourth order Runge-Kutta method. Note that the used Modified Decomposition Method gives the critical Reynolds number for larger values of channel half-angle α while the earlier applied analytical methods which give the solution as an infinite series are supposed to be valid for very small parameters values.

Figures 12 and 13 show the effect of channel half-angle,α, on the velocity profiles in convergent-divergent channels for a fixed Reynolds number. In this case, we can see clearly that the effect of increasing of the channel half-angle, α, is predicted to be similar to the effect of Reynolds number. Indeed, in Fig. 12, we can observe for converging channel, that the increase of the channel half-angle, α, accelerates fluid motion near the wall, while there is a reverse behaviour for diverging channel, as shown in Fig. 13. These behaviours can be explained by the increase of the favourable pressure gradient for converging channel, but for diverging channel, we can see that the separation and backflow may occur for large values of the channel half-angle, α, where the adverse pressure gradient is large.

To illustrate more accuracy of the adopted modified decomposition method, a comparison with numerical and other reported results for several values of $R_e\alpha$ is plotted in Figs. 14 and 15. Indeed, presented results showed an excellent agreement between them which further confirms the validity, applicability and higher accuracy of the Modified Decomposition Method.

Finally, in order to compare velocities profiles in divergent channel (Fig. 15, Tables 3 and 4) between different used methods, it is worth noting that the obtained results by MDM method are reversed and consequently the velocities varies from F = 0 at η=±1 to F = 1 at η=0.

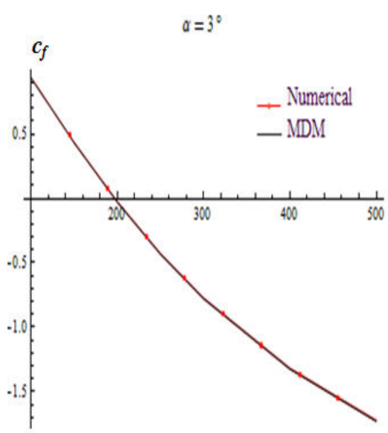


Fig. 10, Skin friction versus Reynolds number for divergent channels (α=3°).

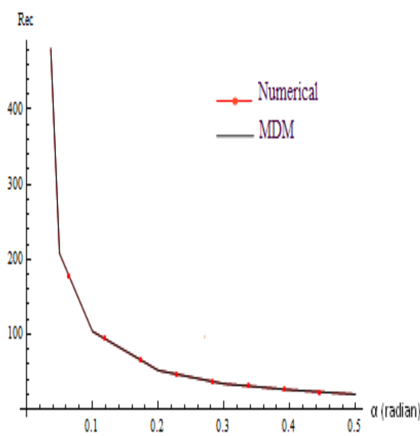


Fig. 11, A plot of the critical Reynolds number R_{ec}, based on the axial velocity, against the half-angle of the channel, α.

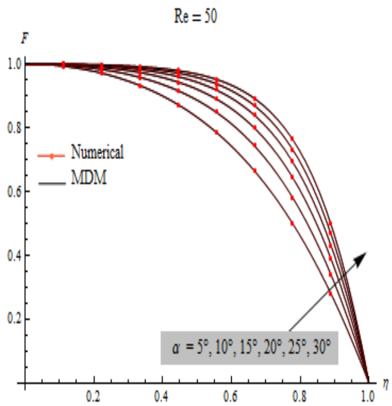


Fig. 12. Effect of channel half-angle α on velocity profiles for convergent channels (Re=50).

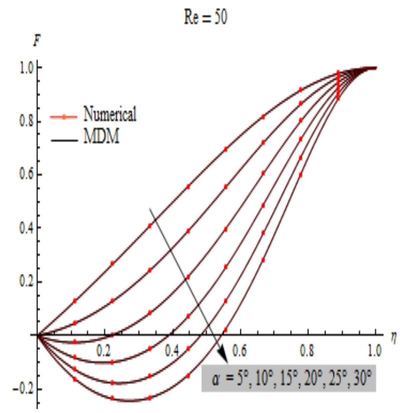


Fig. 13. Effect of channel half-angle α on velocity profiles for divergent channels (Re=50).

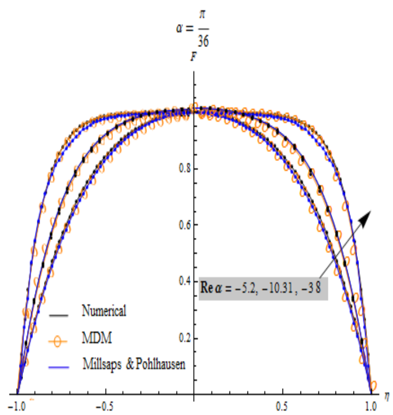


Fig. 14. Comparison between different results for convergent channels-velocity profiles versus $R_e\alpha$.

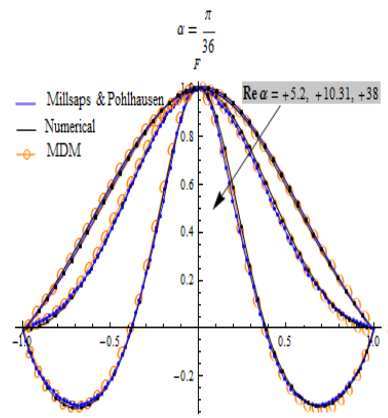


Fig. 15. Comparison between different results for divergent channels-velocity profiles versus $R_e\alpha$.

6. Conclusion

In this paper the nonlinear problem of an incompressible viscous flow between nonparallel plane walls known as Jeffery-Hamel flow is investigated analytically and numerically. Indeed, the third order nonlinear differential equation which governs the Jeffery-Hamel flow has been solved analytically by using a Modified Decomposition Method and numerically via fourth order Runge Kutta method. The principal aim of this study is to obtain an approximation of the analytical solution of the considered problem.

The principal conclusions, which we can draw from this study, are:

- Increasing Reynolds number of convergent flow leads to a flatter profile at the center of channels and consequently the thickness of boundary layer decrease.

- For divergent flow, the effect of increasing Reynolds number is to concentrate the volume flux at the center of channels. In this case, the thickness of boundary layer increases with increasing Reynolds number.
- The increase of the channel half-angle, α, leads to an increase of velocity in convergent channel, while there is a reverse behaviour in velocity profiles for divergent channel.
- For divergent channel, the separation and backflow may occur for higher values of the channel half-angle, α, when the adverse pressure gradient is large.
- Obtained results for dimensionless velocity profiles show an excellent agreement between MDM and numerical solution.
- In comparison with the standard Adomian method, we notice that MDM has a high precision than ADM.
- Modified Decomposition Method gives the solution of the studied problem for larger values of channel half angle α while the earlier applied analytical methods which give the solution as an infinite series are supposed to be valid only for small parameters values.
- The adopted Modified Decomposition Method gives a computationally efficient formulation with an acceleration of convergence rate. Indeed, this method is accurate, efficient and highly recommended to solve nonlinear physical problems because it gives the solution as a fast rapidly convergent power series.

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