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The Probability that an Element of Metacyclic 2-Groups of Positive Type Fixes a Set

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Graphical abstract

$$P_G(S)=\frac{\left|\{(g,s)\mid gs=s:g\in G,s\in S\}\right|}{|S||G|}$$

Abstract

In this paper, G is a metacyclic 2-group of positive type of nilpotency of class at least three. Let Ω be the set of all subsets of all commuting elements of G of size two in the form of (a,b), where a and b commute and each of order two. The probability that an element of a group fixes a set is considered as one of the extensions of the commutativity degree that can be obtained by some group actions on a set. In this paper, we compute the probability that an element of G fixes a set in which G acts on a set, Ω by conjugation.

Keywords: Commutativity degree; metacyclic groups; group action

Abstrak

Dalam kertas kerja ini, G adalah satu kumpulan-2 metakitaran jenis positif dengan kelas nilpoten sekurang-kurangnya tiga. Katalah Ω adalah set bagi semua subset untuk semua unsur kalis tukar tertib bagi G dengan saiz dua dalam bentuk (a,b) di mana a dan b adalah kalis tukar tertib dan setiapnya adalah berperingkat dua. Kebarangkalian bahawa suatu unsur dalam satu kumpulan menetapkan suatu set dianggap sebagai salah satu daripada perluasan bagi darjah kekalisan tukar tertib yang boleh didapati daripada beberapa tindakan kumpulan dalam suatu set. Dalam kertas kerja ini, kami mengira kebarangkalian bahawa satu unsur bagi G menetapkan satu set yang mana G bertindak secara kekonjugatan ke atas Ω.

Kata kunci: Darjah kekalisan tukar tertib; kumpulan metakitaran; tindakan kumpulan

1.0 INTRODUCTION

The commutativity degree is a concept that is used to determine the abelianness of a group. It is denoted by $P(G)$. The definition of the commutativity degree is given as follows.

Definition 1.1[1] Let G be a finite non-abelian group. Suppose that x and y are two random elements of G. The probability that two random elements commute is given as follows.

$$P(G)=\frac{\left|\{(x,y)\in G\times G:xy=yx\}\right|}{|G|^2}.$$

The first investigation of the commutativity degree of symmetric groups was done in 1968 by Erdos and Turan[2]. Few years later, Gustafson[3] and MacHale[4] found an upper bound for the commutativity degree of all finite non-abelian groups in which $P(G)\le\frac{5}{8}$. The concept of the commutativity degree has been generalized and extended by several authors. In this paper, we use one of these extensions, namely the probability that a group element fixes a set. This probability was firstly introduced by Omer *et al*[5] in 2013.In this research, we apply it to the metacyclic 2-groups of positive type of nilpotency class two and class at least three.

In 1975, a new concept was introduced by Sherman[6], namely the probability of an automorphism of a finite group fixes an arbitrary element in the group. The definition of this probability is given as follows:

Definition 1.2[6] Let G be a group. Let X be a non-empty set of G (G is a group of permutations of X). Then the probability of an automorphism of a group fixes a random element from X is defined as follows:

$$P_G(X) = \frac{\left|\left\{(g,x) \mid gx = x, \forall g \in G, x \in X\right\}\right|}{|G||X|}.$$

In 2011, Moghaddam[7] explored Sherman's definition and introduced a new probability, which is called the probability of an automorphism fixes a subgroup element of a finite group, the probability is stated as follows:

$$P_{A_G}(H,G) = \frac{\left|\left\{(\alpha,h) \mid h^\alpha : h \in H, \alpha \in A_G\right\}\right|}{|H||G|},$$

where A_G is the group of automorphisms of a group G. It is clear that if $H = G$, then $P_{A_G}(G,G) = P_{A_G}$.

Next, we state some basic concepts that are needed in this paper.

Definition 1.3[8] A group G is called a metacyclic if it has a cyclic normal subgroup H such that the quotient group G/H is also cyclic.

Definition 1.4[9] Let G be a finite group. Then, G acts on itself if there is a function $G \times G \to G$, such that

$i.(gh)x = g(hx), \forall g,h,x \in G.$

$ii.1_G x = x, \forall x \in G.$

Definition 1.5[10] Let G be any finite group and X be a set. G acts on X if there is a function $G \times X \to X$, such that

(1) $(gh)x = g(hx), \forall g,h \in G, x \in X$

(2) $1_G x = x, \forall x \in G.$

Next, we provide some concepts related to metacyclic p-groups. Throughout this paper, p denotes a prime number.

In 1973, King[11] gave the presentation of metacyclic p-groups, as given in the following:

$$G \cong \langle a, b : a^{p^{\alpha_1}} = 1, b^{p^{\alpha_2}} = a^n, [b,a] = a^m \rangle, \text{ where } \alpha_1, \alpha_2 \geq 0, m > 0, n \leq p^{\alpha_1}, p^{\alpha_1} \mid n(m-1).$$

In 2005, Beuerle [12] separated the classification into two parts, namely for the non-abelian metacyclic p-groups of class two and class at least three. Based on[12], the metacyclic p- groups of nilpotency class two are then partitioned into two families of non-isomorphic p-groups stated as follows:

$G \cong \langle a,b : a^{p^\alpha} = 1, b^{p^\beta} = 1, [a,b] = a^{p^{\alpha-\gamma}} \rangle$, where $\alpha,\beta,\gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$ and $\beta \geq \gamma \geq 1$.

$G \cong Q_8.$

Meanwhile, the metacyclic p-groups of nilpotency class of at least three (p is an odd prime) are partitioned into the following groups:

$G \cong \langle a,b : a^{p^\alpha} = 1, b^{p^\beta} = 1, [b,a] = a^{p^{\alpha-\gamma}} \rangle$, where $\alpha,\beta,\gamma \in \mathbb{N}, \alpha - 1\gamma < 2$ and $\alpha \leq \beta$.

$G \cong \langle a,b : a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\omega}}, [b,a] = a^{p^{\alpha-\gamma}} \rangle$, where $\alpha, \beta,\gamma,\varepsilon \in \mathbb{N}, \alpha - 1\gamma < 2\gamma, \alpha \leq \beta$ and $\alpha \leq \beta + \omega$.

Moreover, metacyclic p-groups are also classified into two types, namely negative and positive[12]. The following notations for these two types which are used in this paper are represented as follows:

$G(\alpha,\beta,\omega,\gamma,\pm) \cong \langle a,b : a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\omega}}, [b,a] = a^t \rangle$, where $\alpha,\beta,\gamma,\omega \in \mathbb{N}, t = p^{\alpha-\gamma} \pm 1.$

If $t = p^{\alpha-\gamma} - 1$, then the group is called a metacyclic of negative type and it is of positive type if $t = p^{\alpha-\gamma} + 1$. Thus, $G(\alpha,\beta,\in,\gamma,-)$ denotes the metacyclic group of negative type, while $G(\alpha,\beta,\in,\gamma,+)$ denotes the positive type. These two notations are shortened to $G(p,+)$ and $G(p,-)$ respectively[11,12].

The scope of this paper is only for the metacyclic p-groups of positive type.

Theorem 1.1[12] Let G be a metacyclic 2-group of positive type of nilpotency class of at least three. Then G is isomorphic to one of the following types:

(1) $G \cong \langle a,b : a^{2^\alpha} = b^{2^\beta} = 1, [b,a] = a^{2^{\alpha-\gamma}} \rangle, \alpha,\beta,\gamma \in \mathbb{N}, \beta \geq \gamma$ $1 + \gamma < \alpha < 2\gamma.$

(2) $G \cong \langle a,b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\omega}}, [b,a] = a^{2^{\alpha-\gamma}} \rangle, 1 + \gamma < \alpha < 2\gamma,$ $\gamma \leq \beta, \alpha \leq \beta + \omega.$

Definition 1.6[13] Let G act on a set S, and $x \in S$. If G acts on itself by conjugation, the orbit $O(x)$ is defined as follows:

$$O(x) = \left\{y \in G : y = axa^{-1} \text{ for some } a \in G\right\}.$$

In this case $O(x)$ is also called the conjugacy classes of x in G. Throughout this paper, we use $K(G)$ as a notation for the number of conjugacy classes in G.

This paper is structured as follows: Section 1 provides some fundamental concepts of group theory which are used in this paper. In section 2, we state some of previous works, which are related to the commutativity degree, in particular related to the probability that a group element fixes a set. The main results are presented in Section 3.

2.0 PRELIMINARIES

In this section, we provide some previous works related to the commutativity degree, more precisely to the probability that an element of a group fixes a set.

Recently, Omer *et al.*[4] extended the commutativity degree by defining the probability that an element of a group fixes a set of size two, given in the following.

Definition 2.1[4] Let G be a group. Let S be a set of all subsets of commuting elements of G of size two. If G acts on S, then the probability of an element of a group fixes a set is defined as follows:

$$P_G(S)=\frac{\left|\{(g,s)\mid gs=s : g\in G, s\in S\}\right|}{|S||G|}.$$

The following result from Omer *et al.*[5] will be used in our proof in the next section.

Theorem 2.1[4] Let G be a finite group and let X be a set of elements of G of size two in the form of (a,b) where a and b commute. Let S be the set of all subsets of commuting elements of G of size two and G acts on S by conjugation. Then the probability that an element of a group fixes a set is given by $P_G(S)=\frac{K(S)}{|S|}$, where $K(S)$ is the number of conjugacy classes of S in G.

Recently, Mustafa *et al.*[14] has extended the work in [4] by restricting the order of Ω. The following theorem illustrates their results.

Theorem 2.2[14] Let G be a finite group and let S be a set of elements of G of size two in the form of (a,b), where a,b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω. Then the probability that an element of a group fixes a set is given by $P_G(\Omega)=\frac{K(\Omega)}{|\Omega|}$, where $K(\Omega)$ is the number of conjugacy classes of Ω in G.

■3.0 MAIN RESULTS

This section provides our main results. First, we provide the following theorem that illustrates the case when $P_G(\Omega)$ is equal to one. Then, the probability that an element of metacyclic2-groups of positive type of nilpotency class at least three fixes a set is computed.

Theorem 3.1 Let G be a finite non-abelian group. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation, then $P_G(\Omega)=1$ if and only if all commuting elements a and b are in the center of G.

Proof First, suppose $P_G(\Omega)=1$. Then by Theorem 2.2, $K(\Omega)=|\Omega|$, since G acts on Ω by conjugation, then there exists a function $\Psi: G\times\Omega\to\Omega$ such that $\Psi_g(w)=gwg^{-1}$, where $w\in\Omega$, $g\in G$. Since $K(\Omega)=|\Omega|$, then the conjugacy class of (a,b) is $\text{cl}(a,b)=(a,b), \forall\ (a,b)\in\Omega$. By the conjugation thus $\text{cl}(w)=gwg^{-1}$, $w\in\Omega, g\in G$. Therefore, the conjugacy class of $\text{cl}((a,b))=(gag^{-1},gbg^{-1})$, $g\in G, \omega\in\Omega$. Since $K(\Omega)=|\Omega|$ thus $\text{cl}(a,b)=(a,b)$ for all $(a,b)\in\Omega$. It follows that, $(gag^{-1},gbg^{-1})=(a,b)$ thus $gag^{-1}=a$, $gbg^{-1}=b$. From which we have $ga=ag$, $gb=bg$, $\forall\ g\in G$, $\omega\in\Omega$. Since g and (a,b) are arbitrary elements of G and Ω respectively, then $a,b\in Z(G)$, as claimed. Second, assume that $a,b\in Z(G)$. Since the action here is by conjugation, thus $\text{cl}(w)=gwg^{-1}, w\in\Omega, g\in G$. Thus $\text{cl}(a,b)=(gag^{-1},gbg^{-1})$, since $a,b\in Z(G)$, thus $ag=ga$, $bg=gb, \forall g\in G$. Therefore, $\text{cl}(a,b)=(a,b)$, for all $(a,b)\in\Omega$. Hence, $K(\Omega)=|\Omega|$. Using Theorem 2.2, $P_G(\Omega)=1$, as required.■

In the following two theorems, the probability that a group element fixes a set of metacyclic 2-groups of positive type of nilpotency class at least three is computed. In both theorems, let S be a set of elements of G of size two in the form of (a, b), where a and b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation.

Theorem 3.2 Let G be a finite group of type (1), $G\cong\langle a,b : a^{2^{\alpha}}=b^{2^{\beta}}=1, [b,a]=a^{2^{\alpha-\gamma}}\rangle$, where $\alpha,\beta,\gamma\in\mathbb{N}$, $1+\gamma<\alpha<2\gamma$, $\beta\geq\gamma$. Then $P_G(\Omega)=\frac{2}{|\Omega|}$.

Proof If G acts on Ω by conjugation, then there exists $\Psi: G\times\Omega\to\Omega$ such that $\Psi_g(w)=gwg^{-1}$, $w\in\Omega, g\in G$. The elements of order two in G are $a^{2^{\alpha-1}}$, $b^{2^{\beta-1}}$ and $a^{2^{\alpha-1}}b^{2^{\beta-1}}$. Therefore, the elements of Ω are stated as follows: Two elements in the form of $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}}\right), 0\leq i\leq 2^{\alpha}$, and one element in the form $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}}\right)$. Then $|\Omega|=3$. If G acts on Ω by conjugation then $\text{cl}(w)=gwg^{-1}, w\in\Omega, g\in G$. The conjugacy classes can be described as follows: One conjugacy class in the form of $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}}\right), 0\leq i\leq 2^{\alpha}$, and one conjugacy class in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}}\right)$. Thus there are two conjugacy classes. Using Theorem 2.2, then $P_G(\Omega)=\frac{2}{|\Omega|}$.■

Next, the probability that an element of a group of type (2) fixes a set is computed.

Theorem 3.3 Let G be group of type (2), $G\cong\langle a,b : a^{2^{\alpha}}=1, b^{2^{\beta}}=a^{2^{\alpha-\varepsilon}}, [b,a]=a^{2^{\alpha-\gamma}}\rangle$, $1+\gamma<\alpha<2\gamma, \gamma\leq\beta$, $\alpha\leq\beta+\varepsilon$. Then

$$P_G(\Omega)=\begin{cases}\dfrac{2}{|\Omega|}, & if\ \alpha>\beta+1,\\ 1, & if\ \alpha\le\beta+1.\end{cases}$$

Proof First, let $\alpha>\beta+1$. If G acts on Ω by conjugation, then there exists $\Psi: G\times\Omega\to\Omega$ such that $\Psi_g(w)=gwg^{-1}$, $w\in\Omega$, $g\in G$. Thus, the elements of order two in G are $a^{7\times 2^{\alpha-4}+2^{\alpha-1}}b^{2^{\beta-1}}$, $a^{7\times 2^{\alpha-4}}b^{2^{\beta-1}}$ and $a^{2^{\alpha-1}}$. Hence, the elements of Ω are stated as follows: Two elements are in the form of $(a^{2^{\alpha-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}i}b^{2^{\beta-1}})$, $0\le i\le 1$, and there is only one element in Ω which is in the form of $(a^{7\times 2^{\alpha-4}}b^{2^{\beta-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}}b^{2^{\beta-1}})$, from which it follows that $|\Omega|=3$. Since the action is by conjugation, then the conjugacy classes can be described as follows: One conjugacy class in the form $(a^{2^{\alpha-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}i}b^{2^{\beta-1}})$, $0\le i\le 1$, and one conjugacy class in the form $(a^{7\times 2^{\alpha-4}}b^{2^{\beta-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}}b^{2^{\beta-1}})$. Thus the number of conjugacy classes is two. Using Theorem 2.2, then $P_G(\Omega)=\dfrac{2}{|\Omega|}$. For the second case, namely when $\alpha\le\beta+1$, the elements of order two in G are similar to the elements in the first case. Hence, the elements of Ω are stated as follows: Two elements in the form of $(a^{2^{\alpha-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}i}b^{2^{\beta-1}})$, $0\le i\le 1$, and one element in the form $(a^{7\times 2^{\alpha-4}}b^{2^{\beta-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}}b^{2^{\beta-1}})$. Therefore, $|\Omega|=3$. If G acts on Ω by conjugation, the conjugacy classes can be described as follows: One conjugacy class in the form $(a^{2^{\alpha-1}}, a^{7\times 2^{\alpha-4}}b^{2^{\beta-1}})$, one conjugacy class in the form of $(a^{2^{\alpha-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}}b^{2^{\beta-1}})$ and one conjugacy class in the form $(a^{7\times 2^{\alpha-4}}b^{2^{\beta-1}}, a^{7\times 2^{\alpha-4}+2^{\alpha-1}}b^{2^{\beta-1}})$. It follows that, there are three conjugacy classes. Using Theorem 2.2, then $P_G(\Omega)=1$.

■4.0 CONCLUSION

In this paper, the probability that a group element fixes a set is found for metacyclic 2-groups of positive type of nilpotency class at least three. Besides, the necessary and sufficient condition for all non-abelian groups in which $P_G(\Omega)$ is equal to one under conjugation action is provided.

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