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BUCKLING OF CIRCULAR PLATES WITH AN INTERNAL ELASTIC RING SUPPORT AND OUTER EDGE RESTRAINED AGAINST TRANSLATION

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Abstract

This paper deals with exact solution for elastic buckling of circular plates resting on internal elastic ring support and elastically restrained against translation at the outer edge. The classical plate theory is used in deriving the governing differential equation for a circular plate resting on internal ring support and linearly restrained at the outer edge and subjected to in-plane compressive load. From the present study it is found that there exists a buckling mode switching from axi-symmetric to asymmetric as we vary the radius of internal elastic ring support. In this case, the buckling mode is not axi-symmetric as is generally expected. The plate buckles in an asymmetric mode as the radius of the internal ring support becomes smaller. The cross-over ring support radius varies from 0.1255 to 0.09371 times the plate radius, depending upon the stiffness of the translational spring support at the outer edge. Based on this data, the optimum radius of the internal ring support can be determined. Extensive data is presented in tabular form so that significant conclusions can be arrived at on the influence of translational restraints, Poisson's ratio, and boundary conditions on the buckling behavior of isotropic circular plates. The numerical results obtained from the present study are found to be in close agreement with the previously published data.

Keywords: Circular plate, Buckling, Elastic ring support, Flexible edge, Mode switching.

1. Introduction

The prediction of buckling of structural members restrained laterally is an important topic in the design of various engineering components. In particular, the

Nomenclatures

b	Non-dimensional ring support radius parameter
D	Flexural rigidity of a material, N.mm^2
E	Young's modulus of a material, N/mm^2
h	Thickness of a plate, mm
k	Non-dimensional buckling load parameter
K_{T1}	Translational spring stiffness at outer edge, N/mm
K_{T2}	Translational spring stiffness of internal elastic ring, N/mm
N	Uniform in - plane compressive load, N/mm
R	Radius of a plate, mm
T_{11}	Non-dimensional translational flexibility parameter at outer edge
T_{22}	Non-dimensional translational flexibility parameter of internal elastic ring
v	Poisson's ratio

circular plates with an internal elastic ring supports find applications in aeronautical (instrument mounting bases for space vehicles), rocket launching pads, aircrafts and naval vessels (instrument mounting bases). Based on the Kirchhoff's theory, the elastic buckling of thin circular plates has been extensively studied by many authors after the pioneering work published by Bryan [1]. Since then, there have been extensive studies on the subject covering various aspects such as different materials, boundary and loading conditions [2, 3].

However, these sources only considered axisymmetric cases, which may not lead to the correct buckling load. Introducing an internal *elastic ring* supports may increase the elastic buckling capacity of in-plane loaded circular plates significantly. Laura et al. [4] investigated the elastic buckling problem of the aforesaid type of circular plates, who modeled the plate using the classical thin plate theory. In their study only axisymmetric modes are considered. Kunukkasseril and Swamidas [5] are probably the first to consider *elastic ring* supports. They formulated the equations in general, but presented only the case of circular plate with a free edge. Although the circular symmetry of the problem allows for its significant simplification, many difficulties very often arise due to complexity and uncertainty of boundary conditions. This uncertainty could be due to practical engineering applications where the edge of the plate does not fall into the classical boundary conditions. It is an accepted fact that the condition on a periphery often tends to be in between the classical boundary conditions (free, clamped and simply supported) and may correspond more closely to some form of elastic restraints, i.e., rotational and translational restraints [6-8].

In a recent study, Wang and Wang [6] showed that when the ring support has a small radius, the buckling mode takes the asymmetric form. But they have studied only the circular plate with elastically restrained edge against rotation. The purpose of the present work is to complete the results of the buckling of circular plates with an internal *elastic ring* support and elastically restrained edge against translation by including the asymmetric modes, thus correctly determining the buckling loads.

2. Definition of the Problem

Consider a thin circular plate of radius *R*, uniform thickness *h*, Young's modulus *E*, flexural rigidity *D* and Poisson's ratio *v*. It is subjected to a uniform in-plane load, *N* along its periphery i.e., along the outer edge at *r*=1 (non-dimensional radius of outer edge) and it is supported by an internal *elastic ring* support at *r* = *b* (non-dimensional radius of support), as shown in Fig. 1. The problem at hand is to determine the elastic critical buckling load of a circular plate with an internal *elastic ring* support and elastically restrained edge against translation.

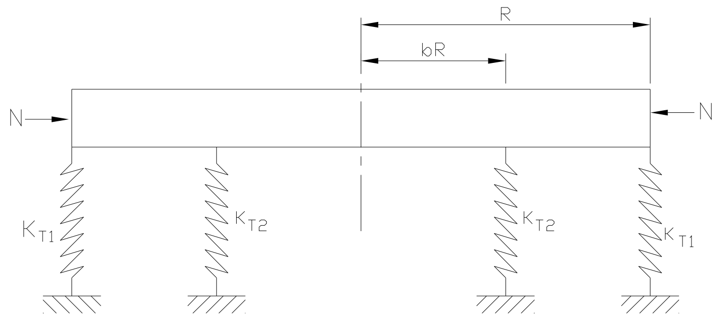

Fig. 1. Buckling of a Circular Plate with an Internal Elastic Ring Support and Restrained Edge against Translation.

3. Formulation of the Problem

The plate is elastically restrained edge against translation at the edge of radius *R* and supported on an internal elastic ring of smaller radius *bR* as shown in Fig. 1. The plate under consideration is subjected to a uniformly distributed in-plane load N , along the outer periphery i.e, at $r=1$ and supported on elastic ring support at $r=b$, as shown in Fig. 1. Let subscript I denote the outer region $b \le \bar{r} \le 1$ and the subscript II denote inner region $0 \le \bar{r} \le b$. Here, all lengths are normalized by *R*. Using the classical thin plate theory (Kirchhoff' theory), the governing fourth order differential equation for elastic buckling of an annular plate may be expressed in polar coordinates (r,θ) as

$$D\nabla^4 w + N\nabla^2 w = 0 \tag{1}$$

where *w* is the lateral displacement, *N* is the uniform compressive load at outer edge. After normalize lengths by the radius of the plate *R*, Eq. (1) can be written as

$$D\nabla^4 \overline{w} + k^2\nabla^2 \overline{w} = 0 \tag{2}$$

$$\nabla^2(\nabla^2 + k^2)\overline{w} = 0 \tag{3}$$

where the Laplacian operator, $\nabla^2 = \frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2}{\partial \theta^2}$ (4)

where $\bar{r}$ is the radial distance normalized by R . $\overline{D} = Eh^3/12(1-\nu^2)$ is the flexural rigidity, $\overline{w} = w/R$, is normalized transverse displacement of the plate. $k^2 = R^2 N/\overline{D}$ is non-dimensional load parameter. Suppose there are n nodal diameters. In polar coordinates (r,θ) set

$$\bar{w}(\bar{r},\theta)=\bar{u}(\bar{r})\cos(n\theta) \tag{5}$$

Considering the boundness at the origin, the general solution [8] for the two regions is

$$\bar{u}_I(r)=C_1J_n(k\bar{r})+C_2Y_n(k\bar{r})+C_3\bar{r}^n+C_4\begin{Bmatrix}\log\bar{r}\\ \bar{r}^{-n}\end{Bmatrix} \tag{6}$$

$$\bar{u}_{II}(r)=C_5J_n(k\bar{r})+C_6\bar{r}^n \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5) yields the following

$$\bar{w}_I(\bar{r},\theta)=\left[C_1J_n(k\bar{r})+C_2Y_n(k\bar{r})+C_3\bar{r}^n+C_4\begin{Bmatrix}\log\bar{r}\\ \bar{r}^{-n}\end{Bmatrix}\right]\cos(n\theta) \tag{8}$$

$$\bar{w}_{II}(\bar{r},\theta)=\left[C_5J_n(k\bar{r})+C_6\bar{r}^n\right]\cos(n\theta) \tag{9}$$

where top form of the Eq. (9) is used for $n=0$ (symmetric) and the bottom form is used for $n\neq 0$ (asymmetric), C_1, C_2, C_3, C_4, C_5 and C_6 are constants, $J_n(.)$ and $Y_n(.)$ are the Bessel functions of the first and second kinds of order n, respectively.

The boundary conditions at outer region of the circular plate is given by the following expressions

$$\bar{u}_I'(\bar{r})=0 \tag{10}$$

$$V_r(\bar{r})=-K_{T1}\bar{u}_I(\bar{r}) \tag{11}$$

The radial Kirchhoff shear at outer edge is defined as follows

$$V_r(\bar{r})=-\frac{D}{R^3}\left[\bar{u}_I'''(\bar{r})+\bar{u}_I''(\bar{r})-\bar{u}_I'(\bar{r})(n^2(2-\nu))+n^2(1-\nu)\bar{u}_I(\bar{r})\right] \tag{12}$$

Equations (11) and (12) yields the following

$$\left[\bar{u}_I'''(\bar{r})+\bar{u}_I''(\bar{r})-\bar{u}_I'(\bar{r})(n^2(2-\nu))+n^2(1-\nu)\bar{u}_I(\bar{r})\right]=T_{11}\bar{u}_I(\bar{r}) \tag{13}$$

where $T_{11}=\dfrac{K_{T1}R^3}{D}$ (14)

Apart from the elastically restrained edge against translation, there is an internal *elastic ring* support constraint and the continuity requirements of slope and curvature at the support, i.e. $\bar{r}=b$

$$\bar{u}_I(b)=\bar{u}_{II}(b) \tag{15}$$

$$\bar{u}_I'(b)=\bar{u}_{II}'(b) \tag{16}$$

$$\bar{u}_I''(b)=\bar{u}_{II}''(b) \tag{17}$$

$$\bar{u}_I'''(b)=\bar{u}_{II}'''(b)-T_{22}\bar{u}_{II}(b) \tag{18}$$

where $T_{22}=K_{T2}R^3/D$ is the normalized spring constant, K_{T2} of the translational spring.

Boundary conditions at the outer edge at $\bar{r}=1$

Equations (8), (10) and (13) yields the following

$$\left[\frac{k^2}{4}P_2+\frac{k\nu}{2}P_1-\left(\frac{k^2}{2}+\nu n^2\right)J_n(k)\right]C_1+\left[\frac{k^2}{4}Q_2+\frac{k\nu}{2}Q_1-\left(\frac{k^2}{2}+\nu n^2\right)Y_n(k)\right]C_2+$$
$$\left[n((n-1)(1-\nu))\right]C_3+\begin{Bmatrix}\nu-1\\ n((n+1)(1-\nu))\end{Bmatrix}C_4=0 \tag{19}$$

$$\left[\frac{k^3}{8}P_3+\frac{k^2}{4}P_2-\frac{k}{2}\left(\frac{3}{4}k^2+n^2(2-\nu)+1\right)P_1+\left(n^2(3-\nu)-\frac{k^2}{2}-T_{11}\right)J_n(k)\right]C_1+$$
$$\left[\frac{k^3}{8}Q_3+\frac{k^2}{4}Q_2-\frac{k}{2}\left(\frac{3}{4}k^2+n^2(2-\nu)+1\right)Q_1+\left(n^2(3-\nu)-\frac{k^2}{2}-T_{11}\right)Y_n(k)\right]C_2+$$
$$\left[n^2(n-1)\nu-n^3-T_{11}\right]C_3-\left\{n^2(2-\nu)-n^2(n+1)\nu+n^3-T_{11}\right\}C_4=0 \quad (20)$$

where

$$P_1=J_{n-1}(k)-J_{n+1}(k);P_2=J_{n-2}(k)+J_{n+2}(k);$$
$$P_3=J_{n-3}(k)-J_{n+3}(k);Q_1=Y_{n-1}(k)-Y_{n+1}(k);$$
$$Q_2=Y_{n-2}(k)+Y_{n+2}(k);Q_3=Y_{n-3}(k)-Y_{n+3}(k);$$

Continuity requirements at the support, i.e, $\bar{r}=b$

Equations (8), (9) and (15)-(16) yield the following

$$J_n(kb)C_1+Y_n(kb)C_2+b^nC_3+\begin{Bmatrix}\log b\\ b^{-n}\end{Bmatrix}C_4-J_n(kb)C_5-b^nC_6=0 \quad (21)$$

$$\frac{k}{2}P_1'C_1+\frac{k}{2}Q_1'C_2+nb^{n-1}C_3+\begin{Bmatrix}\frac{1}{b}\\ -nb^{-n-1}\end{Bmatrix}C_4-\frac{k}{2}P_1'C_5-nb^{n-1}C_6=0 \quad (22)$$

$$\frac{k^2}{4}(P_2'-2J_n(kb))C_1+\frac{k^2}{4}(Q_2'-2Y_n(kb))C_2+n(n-1)b^{n-2}C_3-$$
$$\begin{Bmatrix}\frac{1}{b^2}\\ n(n+1)b^{-n-2}\end{Bmatrix}C_4-\frac{k^2}{4}(P_2'-2J_n(kb))C_5-n(n-1)b^{n-2}C_6=0 \quad (23)$$

$$\frac{k^2}{8}(P_3'-3P_1')C_1+\frac{k^2}{8}(Q_3'-Q_1')C_2+n(n-1)(n-2)b^{n-3}C_3+$$
$$\begin{Bmatrix}\frac{2}{b^3}-n(n+1)(n+2)b^{-n-3}\end{Bmatrix}C_4-\left[\frac{k^2}{8}(P_3'-3P_1')-T_{22}J_n(kb)\right]C_5-$$
$$\left[n(n-1)(n-2)b^{n-3}-T_{22}b^n\right]C_6=0 \quad (24)$$

where

$$P_1'=J_{n-1}(kb)-J_{n+1}(kb);P_2'=J_{n-2}(kb)+J_{n+2}(kb);P_3'=J_{n-3}(kb)-J_{n+3}(kb);$$
$$Q_1'=Y_{n-1}(kb)-Y_{n+1}(kb);Q_2'=Y_{n-2}(kb)+Y_{n+2}(kb);Q_3'=Y_{n-3}(kb)-Y_{n+3}(kb);$$

The top form of Eqs. (19), (20) and (21)-(24) are used for $n = 0$ (axisymmetric buckling) and the bottom form is used for $n \neq 0$ (asymmetric buckling).

4. Solution

For the given values of n, ν, b, T_{11}, and T_{22} the above set of equations gives an exact characteristic equation non-trivial solution of the coefficients C_1, C_2, C_3, C_4, C_5 and C_6. For non-trivial solution, the determinant of $[C]_{6x6}$ must vanish. The problem is solved using Mathematica, computer software with symbolic capabilities.

5. Results and Discussion

The code developed in Mathematica is used to determine the buckling load parameter for any range of translational constraints. The findings are presented in both tabular and graphical form. The buckling loads are calculated for various internal ring support radii *b*, and translational spring stiffness parameter. The buckling load parameters for *axisymmetric* and *asymmetric* modes for various values of translational stiffness parameters of elastic ring support, T_{22} by keeping translational spring stiffness parameter, T_{11} constant, is presented in Table 1.

Figure 2 shows the variation of buckling load parameter, *k* versus elastic ring support radius parameter, *b* for translational spring stiffness parameter of an elastic ring support (T_{22} =100) by keeping translational spring stiffness parameter constant (T_{11}=1000). It is observed from Fig. 2 that the *n*=0 axisymmetric solution weaves with *n*=1 asymmetric solution. When *b*=0, the untethered plate buckles axisymmetrically with $k = 2.05171$. When *b* increased beyond 0.125 ($k = 3.71692$), the $n = 1$ asymmetric mode gives the correct lower buckling load. This persists until $b = 0.5161$ ($k = 5.13575$) where the $n = 0$ mode again determines the buckling load. It is observed from the Fig. 2 that the buckling is governed by the *axisymmetric* mode $n = 0$, when $b \le 0.125$ and $b \ge 0.5161$. The optimal location of an internal elastic ring support (b_{opt}) is 0.5161 and the corresponding buckling load parameter (k_{opt}) is 5.13575. For this case, an internal elastic ring support, when placed at an optimal position, can increase the buckling load capacity by 34%.

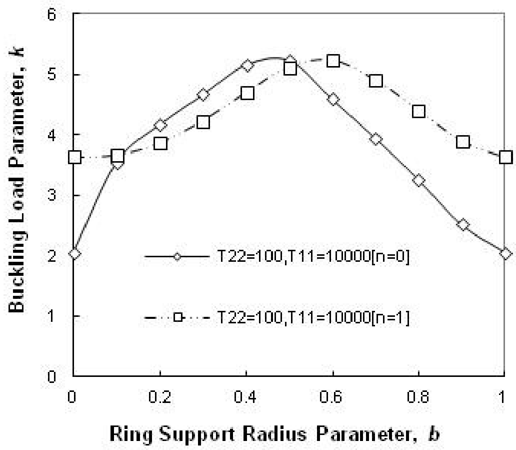

Fig. 2. Buckling Load Parameter versus Internal Elastic Ring Support Radius for Various Values of T_{22} =100 when T_{11}=1000.

Figure 3 shows the variation of buckling load parameter k, with respect to the internal elastic ring support radius *b*, for translational spring stiffness parameter of an elastic ring support (T_{22} =10^{16}) by keeping translational spring stiffness parameter constant (T_{11}=1000). It is observed from the Fig. 3, that for a given value of T_{22} by keeping T_{11} constant, the curve is composed of two segments. This is due to the switching of buckling modes. For a smaller internal elastic ring support radius *b*, the plate buckles in an asymmetric mode (i.e., *n* = 1). In this segment (as shown by dotted lines in Fig. 3) the buckling load decreases as *b* decreases in value. For larger internal elastic ring support radius b, the plate buckles in an axisymmetric mode (i.e., $n = 0$).

In this segment (as shown by continuous lines in Fig. 3) the buckling load increases as b decreases up to a peak point corresponds to maximum buckling load and thereafter decrease as b decreases in value as shown in Fig. 3. The cross over radius is 0.09371 (k=4.508325). It is observed from the Fig. 3 that the *asymmetric* buckling mode dominates small elastic ring support radius parameter b, situation i.e., $b \geq 0.09371$. When $b \geq 0.09371$ the $b = 0$ axisymmetric mode gives the correct lower buckling load. The optimal location of an internal elastic ring support (b_{opt}) is 0.48980 and the corresponding buckling load parameter (k_{opt}) is 5.363089. Similarly for this case, an internal elastic ring support, when placed at an optimal position, can increase the buckling load capacity by 40%.

Table 1. Buckling (for Axisymmetric and Asymmetric Modes) Load Parameters for Different Values of Translational Stiffness of Elastic Ring Parameter T_{22} when T_{11}=1000 and ν = 0.3.

b	T_{22} =100, T_{11}=1000		T_{22} =10^{16}, T_{11}=1000	
	$n = 0$	$n = 1$	$n = 0$	$n = 1$
0	2.05171	3.62463	4.40403	3.81036
0.1	3.54546	3.66107	4.52668	4.55422
0.2	4.17149	3.86098	4.77366	5.05184
0.3	4.66723	4.22897	5.07429	5.60533
0.4	5.16055	4.68984	5.33153	6.22488
0.5	5.23921	5.11781	5.36541	6.7861
0.6	4.5961	5.22724	5.1224	6.85729
0.7	3.94715	4.90345	4.7666	6.43801
0.8	3.25764	4.40407	4.40967	5.93723
0.9	2.50951	3.89459	4.06498	5.45838
1	2.04882	3.62463	2.04912	3.62512

It is observed that the effect of the supporting elastic ring is absent if the ring radius b is one, because at b=1 the outer fixed edge dominates.

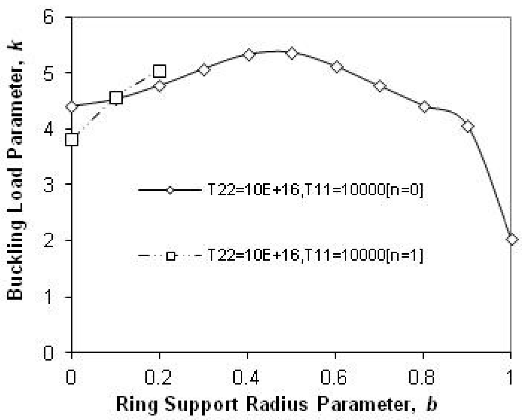

Fig. 3. Buckling Load Parameter versus Internal Elastic Ring Support Radius for various values of T_{22} (=10^{16}) when T_{11}=1000.

Similarly the effect of the internal elastic ring support is negligible for b = 0. The influence of stiffness parameter of elastic ring support is not very much on buckling load at lower values of T_{22}. And also it is observed that the influence of stiffness parameter of elastic ring support is more on buckling load at higher values of T_{22}. The buckling loads increases with increase in value of T_{22}. The results of this kind were scarce in the literature. However, the results are compared for the following cases:

(i) When $T_{22}\rightarrow\infty$, i.e., when the support is rigid, the buckling is governed by the asymmetric mode n=1 when $b \leq 0.152$ and this is in well agreement with the results of Wang and Wang [6]. When $T_{22}\rightarrow\infty$, i.e., rigid ring support, the optimum location is at a radius of b=0.2663, with a buckling load of k=7.0156 that is in well agreement with the results of Wang and Wang [6].

(ii) When $T_{11} \rightarrow \infty$ and $T_{22} \rightarrow \infty$ i.e., when the support is rigid and simply supported edge at the boundary, the buckling load parameters are compared with those obtained by Wang et al. [9] as shown in Table 2 and the results are in good agreement.

Table 2. Comparison of Buckling Load Parameter k, with Wang et al. [9] for Simply Supported Edge for $T_{11}\rightarrow\infty$ and ν = 0.3.

b	Wang et al. [9]	Present
0.1	4.5235	4.52341
0.2	4.7702	4.77018
0.3	5.071	5.07091
0.4	5.3296	5.32964
0.5	5.3666	5.36659
0.6	5.1261	5.12606
0.7	4.7727	4.77266
0.8	4.4215	4.42141
0.9	4.1063	4.10629
0.99	3.8573	3.85742

6. Conclusions

The paper introduced a Mathematica code for calculation of buckling load. The buckling problem of circular plates with an elastically restrained edge against translation has been solved analytically. Also the buckling loads are given for various translational spring stiffness parameter T_{11} at the edges that simulate the translational restraints where $T_{11} \rightarrow \infty$ represents a simply supported edge to free edge where $T_{11} \rightarrow \infty$. It has been observed that the buckling mode switches from an asymmetric mode to an axisymmetric mode at a particular internal ring radius parameter. This cross over radius depends on the translational spring stiffness parameters (T_{11} and T_{22}). The cross-over radius varies from b = 0.1255 for T_{22} =100 to b = 0.09371 for T_{22}=10^{16} by keeping T_{11}=10000. Two-dimensional plots are drawn for a wide range of translational constraints. In this paper the characteristic equations are exact; therefore the results can be calculated to any accuracy. These exact solutions can be used to check numerical or approximate results. Comparison of studies demonstrates the accuracy and stability of the present work. The tabulated buckling results are useful to designers in vibration control, structural design and related industrial applications.

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